

Unit 9

Mathematical language and proof

Introduction

During your mathematical studies, you will have seen many examples of mathematical statements. Some statements are simple and easy to check. For example, consider the following statement:

There are exactly four prime numbers between 10 and 20.

You can easily show that this statement is true, since the prime numbers between 10 and 20 are 11, 13, 17 and 19. Other statements are much more complicated and it is less clear how they can either be shown to be true or shown to be false. A **proof** is a logical argument establishing that a statement is true; the main purposes of this unit are to help you to write proofs yourself and to help you to follow proofs that other people have written.

Important true mathematical statements are often called *theorems* and some of these have become very famous. It is likely that one of the first theorems that you met was Pythagoras' theorem. This concerns a right-angled triangle, as shown in Figure 1.

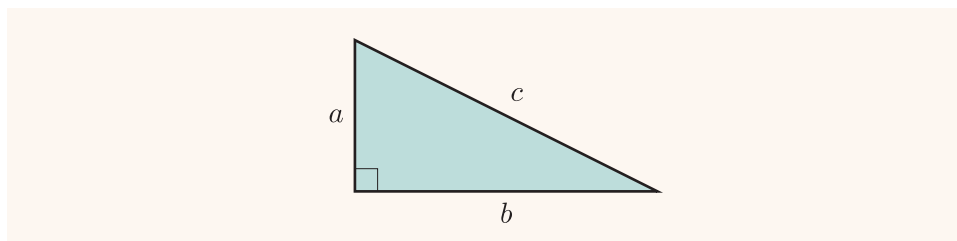


Figure 1 A right-angled triangle

Pythagoras' theorem states that

If a right-angled triangle has sides of length a , b and c , with c being the length of the hypotenuse, then

$$a^2 + b^2 = c^2.$$

For example, if a triangle has sides of length 3 and 4 with a right angle in between, then its hypotenuse has length 5, since

$$\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

This is illustrated in Figure 2.

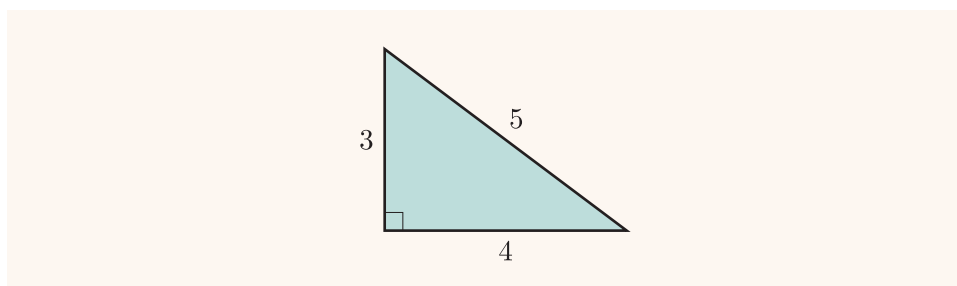


Figure 2 A triangle with sides of length 3, 4 and 5

There are several different proofs of Pythagoras' theorem – some are based on geometric reasoning and others are based on algebraic reasoning. Here is one of the simplest proofs of Pythagoras' theorem. Suppose that you have a right-angled triangle with sides of length a , b and c , where c is the length of the hypotenuse. You can arrange four copies of the triangle as shown in Figure 3.

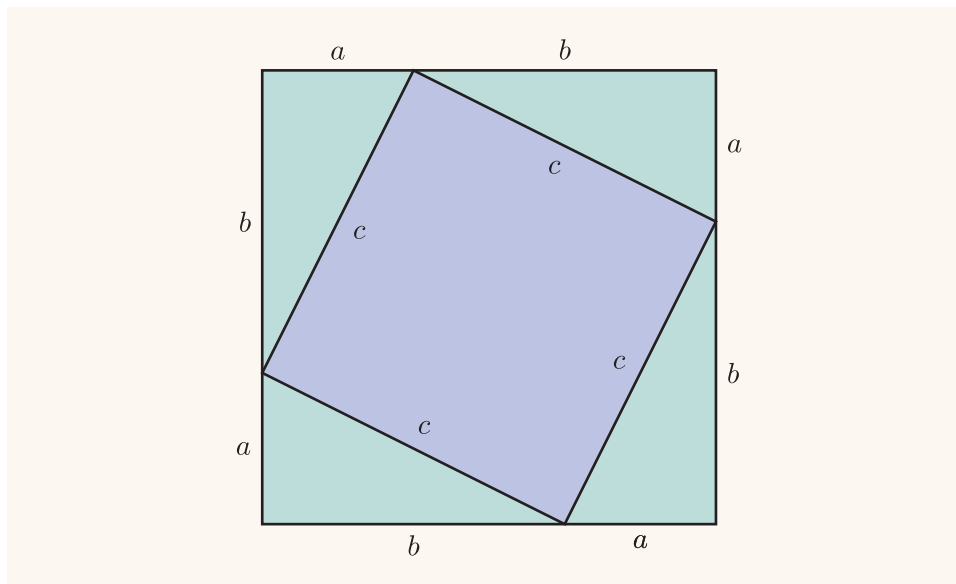


Figure 3 An arrangement of four copies of a right-angled triangle

The area of the large square must equal the sum of the areas of the small square and the four triangles. Thus

$$(a + b)^2 = c^2 + 4 \times \frac{1}{2}ab;$$

that is,

$$a^2 + 2ab + b^2 = c^2 + 2ab$$

and hence

$$a^2 + b^2 = c^2.$$

The structure of this proof is typical of many that you will meet in this unit. We began with a statement that we knew to be true (about the areas of the shapes in Figure 3) and made a sequence of logical deductions until we arrived at the desired conclusion. Learning how to construct and understand proofs of this type will help you with your studies of many different areas of mathematics.

One of the most famous theorems in number theory is *Fermat's last theorem*. You may recall from Unit 3 that this is about equations of the form

$$a^n + b^n = c^n,$$

where a , b , c and n are integers. If $n = 2$ then there are many choices of a , b and c that satisfy the equation above. For example,

$$3^2 + 4^2 = 5^2,$$

as you saw above, and

$$5^2 + 12^2 = 13^2.$$

Fermat's last theorem is about what happens when $n \geq 3$. It states that

If n is an integer greater than 2, then it is impossible to find positive integers a , b and c that satisfy

$$a^n + b^n = c^n.$$

Pierre de Fermat was a lawyer by profession. He was also an amateur mathematician who made notable contributions to many areas of mathematics – in particular, to number theory. His statement of the theorem now known as Fermat's last theorem was discovered after his death written in the margin of a book, together with a statement that the margin was too small to include the proof of the result.

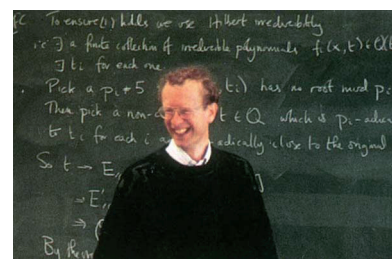


Pierre de Fermat (c. 1601–65)

Several mathematicians attempted to prove Fermat's last theorem without success and it was finally proved by Andrew Wiles over three hundred years after Fermat claimed that it was true. Andrew Wiles himself took many years to prove this result and there were several times when he thought that he had a proof and then discovered a mistake.

The picture in the margin shows Andrew Wiles announcing his proof at the Isaac Newton Institute in Cambridge in 1993. A mistake was subsequently found in the proof but this was fixed after further hard work. Proving any mathematical result, however small, can be very satisfying, as you should soon begin to see for yourself.

In this unit, we will concentrate on results that are easier to prove than Fermat's last theorem! You will, however, see how to prove *Fermat's little theorem*, which you met in Unit 3. You will meet some of the techniques that are often used to establish whether a mathematical statement is true or false. In particular, you will see that often the proof of a statement is constructed from a sequence of simpler statements that are easier to verify. You will get lots of practice at working with statements and this will help you to understand the proofs of theorems that you meet elsewhere and also give you more confidence in proving results yourself. Getting to grips with the proof of a theorem sometimes helps you to understand the theorem better, and to have more confidence in its use.



Andrew Wiles (1953–)

1 Mathematical statements

The key concept in this unit is that of a *mathematical statement*.

Mathematical statements form the building blocks of theorems and proofs, so having a good understanding of how to work with them will help you to read mathematics more easily and write down your own mathematical thoughts and ideas more clearly.

1.1 What are mathematical statements?

Before you can work with mathematical statements, you need a good understanding of what a mathematical statement is. In mathematics, a **statement** is an assertion that is either true or false. The following are examples of statements.

1. All apples are red.
2. Christmas day in 2050 will be a Tuesday.

An example of an assertion that is not a statement is

1000 is a large number.

This is not precise enough to be either true or false.

A **mathematical statement** is a statement about mathematical objects. Here are some simple examples of mathematical statements.

1. 17 is an even number.
2. $5 + 5 = 10$.

Activity 1 Understanding simple mathematical statements

State whether each of the following mathematical statements is true or false.

- (a) $2 + 2 = 3$. (b) The equation $2x - 1 = 0$ has solution $x = \frac{1}{2}$.
 (c) $21 = 7 \times 3$. (d) 25 is a prime number.

The examples of mathematical statements in Activity 1 are particularly simple. In general, it may not be obvious whether a mathematical statement is true or false. Here are some examples of more sophisticated mathematical statements. (Throughout this unit, the phrase ‘is a multiple of’ that appears in the second statement always means ‘is an integer multiple of’.)

1. If n is an odd integer, then n^2 is an odd integer.
2. $n^2 - 2$ is not a multiple of 5, for all $n \in \{0, 1, 2, 3, 4\}$.
3. The function $f(x) = x^2 + 1$ is one-to-one.
4. There are infinitely many prime numbers.
5. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, for all $n \in \mathbb{N}$.

Notice that several of these statements are expressed in terms of a variable n that can take certain integer values. In these statements, the values that n can take are specified by using the ‘belongs to’ symbol $\in$. For example, in the second statement the values of n are those that belong to the set $\{0, 1, 2, 3, 4\}$, and in the final statement they are those that belong to the set $\mathbb{N}$, which is the set of natural numbers $\{1, 2, 3, \dots\}$.

Sometimes the set of values that a variable can take isn’t stated explicitly in a statement and you have to decide what is intended. You’ll see examples of this later in the unit.

It turns out that all of the five statements above apart from the third one are true. During your study of this unit, you will meet a variety of methods that will enable you to produce careful arguments to establish the truth, or otherwise, of all of these statements.

1.2 The language of mathematical statements

There are many words that are used to describe different types of mathematical statements. Even if you don’t remember all of these words, it is useful to be aware that there are different types of statements.

Conjectures

You may have seen various mathematical statements referred to as *conjectures*. Mathematicians use the word **conjecture** to describe a statement that seems likely to be true but has not been proved to be true. One of the most famous conjectures is the *twin prime conjecture*, which is the following statement.

There are infinitely many prime numbers p such that $p + 2$ is also a prime number.

A pair of numbers p and $p + 2$ such that p and $p + 2$ are both prime is known as a **twin prime pair**.

Activity 2 *Identifying twin prime pairs*

Show that there are four twin prime pairs p and $p + 2$ for which $p < 20$.

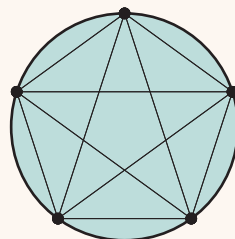
Much progress has been made on understanding the types of twin prime pairs that occur, and examples have been found for extremely large values of p . At the time of writing, however, it is still not known whether the conjecture is true or false.

Mathematicians often make conjectures when they spot a pattern that suggests that a general result may be true. Much mathematical research is motivated by the desire to show that such conjectures are true. Many conjectures are proved relatively easily, while others turn out to be surprisingly difficult to prove, and some turn out to be false!

The following activity asks you to make a conjecture about dividing a disc into smaller regions. (A **disc** is a region whose boundary is a circle.)

Activity 3 *Spotting a pattern*

This activity concerns the maximum number of regions into which a disc can be divided by putting n points on its circumference and drawing a straight line between each pair of points. The diagram below illustrates the case where $n = 5$.



Determine the maximum number of such regions for each value of $n \in \{1, 2, 3, 4, 5\}$, and make a conjecture as to how the number of regions depends on n for each $n \in \mathbb{N}$.

Theorems, lemmas and corollaries

You have already come across several mathematical statements that were referred to as *theorems*. For example, in the introduction you saw Pythagoras' theorem and Fermat's last theorem. In general, the word **theorem** is used for a true mathematical statement of some importance. Once a conjecture has been proved, it may be renamed as a theorem. For example, if the twin prime conjecture were proved, then mathematicians would probably refer to it as the *twin prime theorem*.

Theorems appear in all branches of mathematics and you have already come across them in very different areas of mathematics. As well as the many theorems that you have seen in number theory, you have also met the *binomial theorem* and the *fundamental theorem of calculus*, for example.

When reading mathematics, you may come across other words used to describe true mathematical statements. The word **lemma** is usually used to describe a true statement that is not important enough to be called a theorem. It is often used for a statement that can be used to prove that a more important statement is true. There are no precise rules for deciding whether a statement should be called a lemma or a theorem. Different people may use different words depending on their perception of the importance of the result.

Another word that you may come across is *corollary*. In general, a **corollary** is a mathematical statement whose truth can be established relatively easily by using the fact that a related mathematical statement (often a theorem) is known to be true. For example, the following statement is a corollary of Pythagoras' theorem.

If the two shorter sides of a right-angled triangle both have length a , then the hypotenuse has length $a\sqrt{2}$.

This statement is true since it follows from Pythagoras' theorem that the length of the hypotenuse is

$$\sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2}.$$

You will not see the words 'lemma' or 'corollary' later in this unit, but you may come across them elsewhere.

Another word that appears in a variety of contexts is *result*. The term **result** is often used to refer to the outcome of having carried out some mathematical process. For example, solutions of equations, derived formulas, theorems, lemmas and corollaries are all sometimes referred to as results.

Variable propositions

Mathematical statements are also known as **propositions**. In particular, a mathematical statement that is either true or false depending on the values of one or more variables, such as x or n , is called a **variable proposition**. Here are some examples of variable propositions.

1. x is greater than 0.
2. n is a multiple of 3.

For example, statement 1 above is true when $x = 2$ but false when $x = -1$. In general, many variable propositions are true for some values of the variable and false for other values.

It is sometimes convenient to use the notation $P(x)$ or $P(n)$ to denote a variable proposition that depends on a variable x or n , respectively, especially if you want to refer to particular instances of the proposition, or emphasise the fact that the proposition depends on a particular variable. For example, you could let $P(n)$ denote the variable proposition

n is a multiple of 3.

Then you could say that $P(6)$ is true but $P(5)$ is false.

Sometimes, however, when we do not wish to draw attention to the variable, we use a letter such as P to denote a statement even when the statement includes a variable.

Note that the two variable propositions above don't specify explicitly the kind of values that the variables x and n can take. So you have to decide what the appropriate values are from the context. If we are to decide whether x is greater than 0, or not, then surely x has to take real values. This suggests that the set of values that x can take is the set of real numbers $\mathbb{R}$. Also, since n is a multiple of 3, it must be an integer. So presumably the set of values that n can take is $\mathbb{Z}$.

These conclusions are reinforced by the letters used for the variables. Often (but by no means always) letters near the middle of the alphabet, such as m and n , are used for variables that take integer values, whereas letters near the end of the alphabet, such as x and y , are used for variables that take real values.

Activity 4 Understanding variable propositions

For each of the following variable propositions $P(n)$, give two values of n for which $P(n)$ is true and two values for which $P(n)$ is false.

- (a) n is odd. (b) n is a multiple of 5.

All the variable propositions that you have seen so far in this unit have depended on only one variable. Later you will see examples that depend

on more than one variable. For example, you will consider the variable proposition

m and n are both odd.

The truth of this variable proposition depends on the value of m as well as the value of n . It could be denoted by the notation $P(m, n)$ but we won't need to use this type of notation in this unit.

Existential statements

Another class of mathematical statements that it is useful for you to be aware of is the class of statements known as **existential statements**. The name may sound offputting but it just comes from the fact that such statements include the phrase 'there exists' or another form of words with the same meaning. Here are some examples of existential statements.

1. There exists an even number that is not a multiple of 4.
2. There is a real number that is negative.
3. n^3 is even for some $n \in \mathbb{N}$.
4. The equation $x^3 = 1$ has at least one real solution.

You may have noticed that only the first of these statements contains the phrase 'there exists', and the others use different words with the same meaning. When you write down an existential statement, you can choose whichever form of words you like best. For example, the following four statements all have the same meaning as each other but are written in different ways.

1. *There exists* an even number that is not a multiple of 4.
2. *There is* an even number that is not a multiple of 4.
3. *Some* even numbers are not multiples of 4.
4. There is *at least one* even number that is not a multiple of 4.

You might think that the third statement in the list above has a different meaning to the others, since in everyday language the word 'some' is often used to mean 'several'. In mathematics, however, the word 'some' just means 'at least one' and so all four of these statements do have the same meaning.

You may have come across the special symbol $\exists$ that is used to denote 'there exists' in mathematics. This provides us with a shorthand that allows us to write down mathematical statements more quickly. For example, the statement

the equation $x^3 = 1$ has at least one real solution

can be abbreviated as

$\exists x \in \mathbb{R}$ such that $x^3 = 1$.

The symbol $\exists$, and any words or phrases with the same meaning, such as 'there exists' and 'there are', are known as **existential quantifiers**.

Activity 5 *Recognising existential statements*

Which of the following statements are existential statements?

- (a) There exists a multiple of 3 that is also a multiple of 6.
- (b) 2 is a prime number.
- (c) $\exists x \in \mathbb{R}$ such that $x + 2 > 1$.
- (d) $x + 2 > 1$.
- (e) $2n + 1$ is odd for some $n \in \mathbb{N}$.
- (f) There is at least one real number x such that $x^2 - x > 0$.
- (g) All prime numbers are even.

You can think of an existential statement as a statement that says that a variable proposition is true for at least one value of the variable in a given set. For example, the existential statement in Activity 5(f) says that the variable proposition

$$x^2 - x > 0$$

is true for at least one $x \in \mathbb{R}$.

In general, to show that an existential statement is true, you just need to identify its variable proposition, say $P(x)$, and find one value of x that makes $P(x)$ true. The following example should clarify the method.

Example 1 *Deciding whether an existential statement is true*

Show that the following existential statement is true.

There is an even number that is not a multiple of 4.

Solution

 First identify the variable proposition. 

This statement can be rewritten as

there is an even number n for which $P(n)$ is true,

where $P(n)$ is the variable proposition

n is not a multiple of 4.

 You just need to find one even number n for which $P(n)$ is true. It makes sense to try small values of n first. 

If $n = 2$, then n is even and n is not a multiple of 4; that is, $P(2)$ is true. So the statement is true.

The method of proof used in Example 1 is known as **proof by construction**, so called because it involves finding (or constructing) a particular value of the variable for which the variable proposition is true.

Activity 6 *Deciding whether existential statements are true*

Rewrite each of the following existential statements in a form that involves a variable proposition, and decide whether the existential statement is true or false. For each that is true, give a suitable example to demonstrate this.

- (a) There exists an odd number n such that n^2 is odd.
- (b) Some real numbers are negative.
- (c) There is a real number whose square is negative.

Universal statements

The final class of statements that it is useful for you to be aware of is the class of statements known as **universal statements**. These are statements that include the phrase ‘for all’ or another form of words with the same meaning. Here are some examples of universal statements.

- 1. $x^2 \geq 0$, for all real numbers x .
- 2. Every multiple of 4 is an even number.
- 3. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, for each natural number n .
- 4. All prime numbers are odd.

As with existential statements, there are many equivalent ways of writing a universal statement. The following four statements all have the same meaning as each other but are written in different ways.

- 1. *Every* multiple of 4 is an even number.
- 2. *Any* multiple of 4 is an even number.
- 3. n is even *whenever* n is a multiple of 4.
- 4. n is even *for each* n that is a multiple of 4.

The second statement in the list above uses the word ‘any’. This has two meanings in everyday language – in addition to meaning ‘every’, it can also mean ‘at least one’ as in ‘Have you seen any red cars?’. We try to avoid using the word ‘any’ in mathematical statements if it might lead to confusion. In the statement above, the meaning is clear from the context.

There is one form of words that is often used in statements of this type that does not, at first sight, appear to be similar to those given above. Statements that begin with a phrase like ‘there are no...’ or ‘there does not exist...’ are universal statements because they can be rewritten as statements that include the words ‘for all’. For example, the statement

there is no multiple of 4 that is odd

has the same meaning as the four statements in the second list above.

You may have come across the special symbol $\forall$ that is used to denote ‘for all’ in mathematics. This again provides us with a shorthand that allows us to write down mathematical statements more quickly. For example, the statement

$$x^2 \geq 0, \text{ for all real numbers } x$$

can be abbreviated as

$$x^2 \geq 0, \forall x \in \mathbb{R}.$$

The symbol $\forall$, and any words or phrases with the same meaning, such as ‘for all’, and ‘for every’, are known as **universal quantifiers**.

Activity 7 *Recognising universal and existential statements*

State whether each of the following statements is a universal statement, an existential statement or neither.

- (a) There is a multiple of 5 that is not a multiple of 10.
- (b) Every multiple of 10 is a multiple of 5.
- (c) 10 is a multiple of 5.
- (d) $\exists n \in \mathbb{N}$ such that $2n + 3 = 5$.
- (e) $2n$ is even, $\forall n \in \mathbb{N}$.
- (f) 4 is even.
- (g) There is at least one real number x such that $x + 1 > 0$.
- (h) There is no odd integer n such that n^2 is even.
- (i) n is an even integer.

You can think of a universal statement as a statement that says that a variable proposition is true for every value of the variable in a given set. For example, the universal statement

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2, \forall n \in \mathbb{N}$$

says that the variable proposition

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

is true for all values of n in the set $\mathbb{N}$.

In general, for a universal statement to be true, its underlying variable proposition $P(x)$ must be true for every value of x in the given set. So to show that the universal statement is false you need to find just one value of x in the given set for which $P(x)$ is false. This fact is the topic of the next subsection.

1.3 Counter-examples

As indicated at the end of the previous subsection, in order to show that a universal statement is false, you just need to find *one* value in the given set for which the underlying variable proposition is false. Such a value is called a **counter-example**.

Sometimes it is easy to spot a suitable counter-example. For example, if you were asked to show that the statement

every even number is a multiple of 4

is false, then you would probably spot reasonably quickly that 2 is a counter-example, since 2 is even but is not a multiple of 4. This is sufficient to show that the statement is false. (There are of course many other numbers that you could choose as a counter-example for this statement.)

In other cases a little more effort is required to spot a counter-example.

Example 2 *Using a counter-example*

Show that the following universal statement is false, by giving a counter-example.

$2^n + 1$ is prime for each prime number n .

Solution

 You just need to find one prime number n for which $2^n + 1$ is not prime. (In other words, if $P(n)$ is the variable proposition

$2^n + 1$ is prime,

then you just need to find one prime number n for which $P(n)$ is false.) It's easiest to test small numbers, so try $n = 2$. 

If $n = 2$, then $2^n + 1 = 2^2 + 1 = 4 + 1 = 5$ which is prime. So 2 is not a counter-example.

 The next smallest prime is 3, so try $n = 3$. 

If $n = 3$ then $2^n + 1 = 2^3 + 1 = 8 + 1 = 9$. But $9 = 3 \times 3$ and so 9 is not a prime number. Thus 3 is a counter-example and the statement is false.



‘A single example is enough to disprove a general claim’

There is no general method for finding counter-examples. It makes sense to look for counter-examples that are as simple as possible, and you will often find that repeatedly testing simple values, as in the example above, is a good approach to take. In other cases, you may need to do some thinking to come up with a suitable counter-example. It is often the case that failed attempts to show that a statement is true can suggest a suitable counter-example!

You may recall that in Activity 3 you were asked to consider the maximum number of regions into which a disc can be divided by choosing n points on its circumference and drawing a straight line between each pair of points. You saw that when n is 1, 2, 3, 4 or 5, the maximum number of regions is 1, 2, 4, 8 and 16, respectively, and this suggested the conjecture that, for each $n \in \mathbb{N}$, the maximum number of regions is 2^{n-1} . This, however, is an example of a conjecture that is in fact false! A counter-example can be found by considering $n = 6$, for which it is possible to show (though not easily) that the maximum number of regions is 31.

Activity 8 *Using a counter-example*

Show that the following statement is false, by giving a counter-example.

$n! + 1$ is prime, for each prime number n .

Sometimes a statement that does not initially look like a universal statement may in fact be one, and so it may be possible to show that it is false by finding a counter-example. For example, suppose that you are asked to show that a particular function f is not one-to-one. At first sight you might think that this is a completely different type of problem to showing that a universal statement is false. Notice, however, that by definition the statement

f is one-to-one

has the same meaning as the universal statement

for all a and b in the domain of f , if $a \neq b$ then $f(a) \neq f(b)$.

In order to show that this statement is false (and hence that f is not one-to-one) you just need to find one counter-example. That is, you need to find one pair of values a and b with $a \neq b$ such that $f(a) = f(b)$.

Activity 9 *Using a counter-example to show that a function is not one-to-one*

Show that the following statement is false, by giving a counter-example.

The function $f(x) = x^2 + 1$ is one-to-one.

1.4 Proof by exhaustion

You have just seen that, in some cases, you can use a counter-example to show, fairly easily, that a universal statement is false. You will now meet a relatively straightforward method that you can use to show that some universal statements are *true*. If a universal statement concerns a set with only a small number of elements, then you can show that the statement is true by considering each element of the set in turn. This method is called **proof by exhaustion**.

Example 3 *Using proof by exhaustion*

Show that the following statement is true, by using proof by exhaustion.

$$n^2 - 2 \text{ is not a multiple of 5, for all } n \in \{0, 1, 2, 3, 4\}.$$

Solution

 In order to show that this statement is true by using proof by exhaustion, we must evaluate $n^2 - 2$ for each value of n in turn and show that none of the answers is a multiple of 5. 

If $n = 0$, then $n^2 - 2 = 0 - 2 = -2$, which is not a multiple of 5.

If $n = 1$, then $n^2 - 2 = 1 - 2 = -1$, which is not a multiple of 5.

If $n = 2$, then $n^2 - 2 = 4 - 2 = 2$, which is not a multiple of 5.

If $n = 3$, then $n^2 - 2 = 9 - 2 = 7$, which is not a multiple of 5.

If $n = 4$, then $n^2 - 2 = 16 - 2 = 14$, which is not a multiple of 5.

Thus the statement is true.

Activity 10 *Using proof by exhaustion*

Show that the following statement is true, by using proof by exhaustion.

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2, \text{ for all } n \in \{1, 2, 3, 4\}.$$

In practice, even when a universal statement concerns only a finite number of cases, there may be too many cases to check by hand. Sometimes a computer is needed to consider them all, and if the number of cases is too large then even a computer will be unable to cope. Several theorems in mathematics have been proved (at least in part) by exhaustion with the aid of a computer. The first major theorem to have been proved in this way was the *four colour theorem*. The proof required the use of many thousands of hours of computer time, and gave rise to much controversy about the validity and elegance of such proofs.

The four colour theorem

The four colour theorem concerns maps formed by drawing curves that divide the plane into contiguous regions, as illustrated in Figure 4. It states that at most four colours are needed to colour the regions of any such map so that no two adjacent regions have the same colour.

This theorem was first conjectured to be true around the middle of the nineteenth century, and a ‘proof’ was published by Alfred Kempe (1849–1922) in 1879. However in 1890 Percy Heawood (1861–1955) found an error in Kempe’s proof, and at the same time managed to prove that five colours are enough. Heawood continued to work on the theorem for sixty years – his last paper on it appeared in 1949 – but a complete proof eluded him.

The theorem was eventually proved by Kenneth Appel and Wolfgang Haken in 1976. To prove the theorem, they used a method based on Kempe’s invalid proof. They showed that any map that requires more than four colours to colour it – that is, any counter-example – must contain a portion looking like one of a specific set of 1936 maps. They then used a computer to check that none of the 1936 maps could be part of a *smallest-sized* counter-example. They deduced that no counter-example can exist.

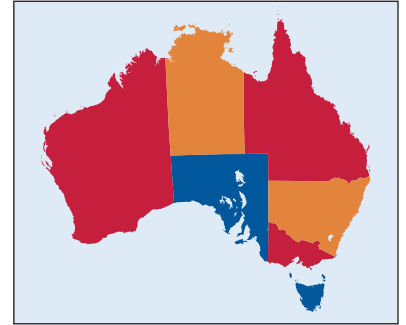


Figure 4 Using four colours to colour a map

Although proof by exhaustion is very effective when there are only a small number of possibilities to consider, it cannot be used when there are infinitely many possibilities. For example, it cannot be used to show that the following statement is true:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2, \text{ for each natural number } n.$$

This is because you cannot check the statement

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

for each natural number n in turn, as there are infinitely many natural numbers. Later in the unit, you will meet a method of proof that can be used very effectively to show that this statement is true.

2 Working with mathematical statements

Since mathematical statements form the building blocks of proofs, it is important for you to become confident in working with them. In this section you will have lots of practice at doing this – in particular, you will gain a better understanding of how the truth of a statement depends on the truth of other related statements. For simplicity, we often use the letters P and Q to denote mathematical statements, both in this section and throughout the rest of the unit.

2.1 Negating statements

Every statement P has a related statement called the **negation** of P , which is true if P is false and false if P is true. For example, the negation of the statement

3 is odd

is the statement

3 is even.

The negation of a statement P is often denoted by $\text{NOT } P$. The following table shows how the truth of $\text{NOT } P$ is related to the truth of P .

P	$\text{NOT } P$
true	false
false	true

This table is a simple example of a *truth table*. In general, a **truth table** is a table that sets out how the truth of one or more statements depends on the truth of one or more other statements. Each of its cells contains either ‘true’ or ‘false’, with sufficient rows to exhaust all possibilities. You will see more examples of truth tables shortly.

You can always write the negation of a statement P as

it is not the case that P ,

although this is often not the simplest way of writing the negation. So, for example, you could have written the negation of the statement ‘3 is odd’ as

it is not the case that 3 is odd

but it is simpler to write the negation as

3 is even.

In the next activity you can practise forming the negations of some simple mathematical statements.

Activity 11 *Writing the negations of mathematical statements*

Write down the negation of each of the following statements, in a simple form.

- (a) 4 is prime.
- (b) $3 < 4$.
- (c) n is even.
- (d) n is a multiple of 5.
- (e) $x + 2 \leq x^2$.

Forming negations of existential and universal statements

The statements in the activity above are all relatively simple. You may find it harder to form the negation of a more complicated statement such as the following:

there exists an even number that is divisible by 4.

It may help to remember that you can always write the negation of a statement P as

it is not the case that P .

So the negation of the statement above can be written as

it is not the case that there exists an even number that is divisible by 4.

However this is quite a complicated way of writing the negation. You can write it more simply as

all even numbers are not divisible by 4.

There are a couple of things worth noting here. First, the original statement is an existential statement (since it starts with ‘there exists’), whereas the negation is a universal statement (since it starts with ‘all’). Second, consider the parts of the statements to which these two phrases refer. The original statement contains the phrase ‘is divisible by 4’, whereas the negation contains the phrase ‘are not divisible by 4’.

In general the following rules hold.

Negating existential and universal statements

- The negation of an existential statement of the form
there exists a ... that is ...
is a universal statement of the form
all ... are not
- The negation of a universal statement of the form
all ... are ...
is an existential statement of the form
there exists a ... that is not

These rules can be made more specific. Consider again the statement
there exists an even number that is divisible by 4.

You can write it as

there exists n in the set of even numbers such that n is divisible by 4.

So its negation is

it is not the case that there exists n in the set of even numbers for which n is divisible by 4.

This statement can be written more simply as

n is not divisible by 4, for all n in the set of even numbers.

The statement and negation above are examples of the following more detailed general rules.

Negating existential and universal statements (detailed rules)

- The negation of an existential statement of the form
there exists $x \in A$ such that $P(x)$ is true
is a universal statement of the form
 $P(x)$ is false for all $x \in A$.
- The negation of a universal statement of the form
 $P(x)$ is true for all $x \in A$
is an existential statement of the form
there exists $x \in A$ such that $P(x)$ is false.

When you are trying to negate a statement, you may find it helpful to use the rules in this box, or you may find it more helpful to use the rules in the previous box. It may depend on how the statement is phrased.

Either way, you may find the following summary helpful.

Negating existential and universal statements (summary)

- Universal quantifiers become existential quantifiers, and vice versa.
- The underlying statement to which a quantifier applies is negated.

If in doubt, when you are trying to negate a statement P , you can always check that the negated statement means the same as

it is not the case that P .

Example 4 *Negating existential and universal statements*

Express concisely the negation of each of the following statements.

- There exists $x \in \mathbb{R}$ such that $x^2 \geq x$.
- Each integer is expressible as the sum of two consecutive integers.
- $n^2 - n$ is even, for all $n \in \mathbb{N}$.

Solution

- (a) Here, an existential quantifier is applied to the proposition ' $x^2 \geq x$ ', so we'll apply a universal quantifier to the negation of this proposition, which is ' $x^2 < x$ '.

The negation is

$$x^2 < x, \text{ for all } x \in \mathbb{R}.$$

- (b) We can write this statement as 'All integers are expressible as the sum of two consecutive integers', so we can use the rules in the first box.

The negation is

there exists an integer that is not expressible as the sum of two consecutive integers.

- (c) Here, a universal quantifier is applied to the proposition ' $n^2 - n$ is even', so we'll apply an existential quantifier to the negation of this proposition, which is ' $n^2 - n$ is odd'.

The negation is

there exists $n \in \mathbb{N}$ such that $n^2 - n$ is odd.



Activity 12 *Negating existential and universal statements*

Express concisely the negation of each of the following statements. In each case, state whether the statement or its negation is true.

- (a) $x^2 \geq 0$, for all $x \in \mathbb{R}$.
- (b) There exists a multiple of 3 that is not even.
- (c) All prime numbers are odd.
- (d) $2n + 1$ is odd, for some $n \in \mathbb{N}$.
- (e) For every natural number n , $n^2 < 32n$.

2.2 Combining statements

Some mathematical statements are combinations of two simpler mathematical statements. When trying to determine whether the combined statement is true, you will often find it helpful to consider the two simpler statements separately.

For example, let P be the statement

1 is odd

and let Q be the statement

2 is odd.

Then P is true, but Q is false.

Now consider the combined statement

1 is odd and 2 is odd.

We denote this statement by P AND Q . It is false since Q is false.

More generally, a combined statement of the form P AND Q is true if *both* of the statements P and Q are true, and is false otherwise, as set out in the following truth table.

P	Q	P AND Q
true	true	true
true	false	false
false	true	false
false	false	false

Activity 13 Working with statements of the form P AND Q

State whether each of the following statements is true or false.

- (a) 1 is odd and 3 is odd.
- (b) 1 is even and 3 is even.
- (c) 2 is odd and 4 is even.
- (d) 2 is even and 4 is odd.

In the activity above, you considered combinations of two simple mathematical statements. You may often want to combine more complicated statements, such as variable propositions. For example, suppose that $P(m)$ is the variable proposition

m is odd

and $Q(n)$ is the variable proposition

n is odd.

You can form the combined statement $P(m)$ AND $Q(n)$ as before to give the following statement:

m is odd and n is odd.

This can be written more simply as

m and n are both odd.

This is a variable proposition with two variables, m and n . It is either true or false depending on the values of m and n .

You can also combine statements in other ways. For example, the two simple statements P and Q that you saw at the beginning of this subsection can be combined to give the following statement:

1 is odd or 2 is odd.

We denote this statement by P OR Q . It is true since P is true.

More generally, a combined statement of the form P OR Q is true whenever at least one of P and Q is true, and is false otherwise, as set out in the following truth table.

P	Q	P OR Q
true	true	true
true	false	true
false	true	true
false	false	false

In everyday English, the meaning of ‘or’ often depends on the context. For example, in the statement

tomorrow I will visit Paul or stay at home,

the word ‘or’ tells us that I will either visit Paul or stay at home, but not both, so ‘or’ is being used in an exclusive sense. Compare this with the use of ‘or’ in the statement

applicants for a card must be unemployed or over 60.

Here, there is nothing to indicate that the applicant cannot be unemployed *and* over 60, so ‘or’ is being used in an inclusive sense.

In mathematics we always use ‘or’ in the inclusive sense.

Activity 14 Working with statements of the form $P \text{ OR } Q$

State whether each of the following statements is true or false.

- (a) 1 is odd or 3 is odd.
- (b) 1 is even or 3 is even.
- (c) 2 is odd or 4 is even.
- (d) 2 is even or 4 is odd.

You can of course also combine variable propositions in this way. For example, suppose that $P(m)$ is the variable proposition

m is odd

and $Q(n)$ is the variable proposition

n is odd.

You can form the combined statement $P(m) \text{ OR } Q(n)$ as before, to give the following statement:

m is odd or n is odd.

This can be written more simply as

at least one of m and n is odd.

This is a variable proposition with two variables, m and n . It is either true or false depending on the values of m and n .

In the next activity you can practise working with different combinations of variable propositions.

Activity 15 *Combining variable propositions*

Let $P(n)$ be the statement

n is a multiple of 2

and $Q(n)$ be the statement

n is a multiple of 3.

Determine whether each of the statements

$P(n)$, $Q(n)$, $P(n)$ AND $Q(n)$, $P(n)$ OR $Q(n)$,

is true or false in each of the following cases.

(a) $n = 6$ (b) $n = 7$ (c) $n = 8$

2.3 Negating combinations of statements

You have now learned how to find the negation of a statement and how to combine statements in different ways. In this subsection you will learn how to form the negations of combined statements and how to write these down as simply as possible.

Consider again the simple statements P and Q , where P is the statement

1 is odd

and Q is the statement

2 is odd.

You began by considering the combined statement P AND Q , which is

1 is odd and 2 is odd.

You can write the negation of this statement as

it is not the case that 1 is odd and 2 is odd,

or, more simply, as

1 is even or 2 is even.

Notice that this is the combined statement

$(\text{NOT } P) \text{ OR } (\text{NOT } Q)$.

(Here the brackets are included to clarify the order in which the operations OR and NOT should be performed.)

In fact, for any two statements P and Q , the negation of P AND Q is always $(\text{NOT } P) \text{ OR } (\text{NOT } Q)$, as you will see in the following activity.

Activity 16 *Using truth tables to check the negation of P AND Q*

- (a) Copy down the truth table for P AND Q from page 246. Then add a further column headed $\text{NOT}(P \text{ AND } Q)$ and fill in each of its cells with either true or false as appropriate.
- (b) Construct a truth table for the column headings P , Q , $\text{NOT } P$, $\text{NOT } Q$, $(\text{NOT } P) \text{ OR } (\text{NOT } Q)$.

Check that the final columns of both tables are identical.

Finding the negation of a statement of the form $P \text{ OR } Q$ is similar, but opposite, to finding the negation of a statement of the form $P \text{ AND } Q$. For example, the negation of the statement

1 is odd or 2 is odd

is the statement

1 is even and 2 is even.

More generally, the negation of the statement $P \text{ OR } Q$ is the statement $(\text{NOT } P) \text{ AND } (\text{NOT } Q)$, as you will see in the following activity.

Activity 17 *Using truth tables to check the negation of $P \text{ OR } Q$*

- (a) Copy down the truth table for $P \text{ OR } Q$ from page 247. Then add a further column headed $\text{NOT } (P \text{ OR } Q)$ and fill in each of its cells with either true or false as appropriate.
- (b) Construct a truth table for the column headings P , Q , $\text{NOT } P$, $\text{NOT } Q$, $(\text{NOT } P) \text{ AND } (\text{NOT } Q)$.

Check that the final columns of both tables are identical.

You have now seen that the following rules hold.

Negating combinations of statements

The negation of $P \text{ AND } Q$ is $(\text{NOT } P) \text{ OR } (\text{NOT } Q)$.

The negation of $P \text{ OR } Q$ is $(\text{NOT } P) \text{ AND } (\text{NOT } Q)$.

Example 5 *Writing the negation of a combination of statements*

Express concisely the negation of the following statement.

Both m and n are even.

Solution

 This statement is of the form P AND Q where P is the statement ‘ m is even’ and Q is the statement ‘ n is even’, so you can use the rule above. 

The negation is

m is odd or n is odd.

This can also be written as

at least one of m and n is odd.

Activity 18 *Negating combinations of statements*

Express concisely the negation of each of the following statements.

- (a) 1 is even or 4 is even. (b) 1 is even and 4 is even.
 (c) At least one of m and n is even. (d) Both of m and n are even.

2.4 Implications and equivalences

The building blocks of many proofs are mathematical statements of two types known as *implications* and *equivalences*. A good understanding of these types of statements, and the differences between them, will help you to write correct proofs and to read other people’s proofs more easily.

Implications

We often use statements of the form ‘if ... then ...’ in real-life situations. For example, you might say the following:

If it is raining, then the children play indoors.

Such a statement is known as an *implication*. Implications are useful because they enable us to deduce the truth of one statement from the truth of another statement. The implication above is built from the two simpler statements ‘it is raining’ and ‘the children play indoors’. If the implication is true, and you know that it is raining, then you can deduce that the children play indoors.

In general, an **implication** is a statement of the form

if P , then Q ,

where P and Q are statements. You can think of it as a ‘promise’ that if P is true, then Q is also true. This means that if in fact P is true but Q is false, then the promise is broken; that is, the implication is false. For example, with the implication about children and rain above, if it is raining and the children are not playing indoors, then the promise is broken and the implication is false.

The implication ‘if P , then Q ’ doesn’t promise anything about what happens when P is false, so if P is false then the promise is not broken; that is, the implication is true. For example, with the implication about children and rain above, if it is not raining and the children are playing indoors, then the promise is not broken and the implication is still true.

In general, the truth or otherwise of an implication ‘if P , then Q ’ is related to the truth or otherwise of the statements P and Q in the way set out in the truth table below.

P	Q	if P , then Q
true	true	true
true	false	false
false	true	true
false	false	true

Let’s look at how this truth table applies to the following mathematical implication:

If n is an even integer, then n is a multiple of 4. (1)

This implication is of the form ‘if $P(n)$, then $Q(n)$ ’ where $P(n)$ and $Q(n)$ are variable propositions:

$P(n)$ means: n is an even integer

$Q(n)$ means: n is a multiple of 4.

Since the truth of $P(n)$ and the truth of $Q(n)$ both depend on the value of n , the truth of the implication also depends on the value of n . In other words, the implication is also a variable proposition. Table 1 shows whether the implication is true or false for a few chosen values of n . The entries in the final column are found by using the truth table above.

Table 1 The truth or otherwise of implication (1) for some values of n

n	$P(n)$	$Q(n)$	if $P(n)$, then $Q(n)$
5	false	false	true
6	true	false	false
8	true	true	true

You may find Table 1 rather disconcerting, because you probably think of implication (1) as a statement that is either definitely true or definitely false, rather than as a variable proposition whose truth depends on the value of the variable n . To help you reconcile these two different possible interpretations of implication (1), let's start by looking at another implication that is a variable proposition:

If n is an even integer, then n^2 is an even integer. (2)

This implication is of the form 'if $P(n)$, then $Q(n)$ ' where:

$P(n)$ means: n is an even integer

$Q(n)$ means: n^2 is an even integer.

Table 2 shows whether the implication is true for a few chosen values of n .

Table 2 The truth or otherwise of implication (2) for some values of n

n	$P(n)$	$Q(n)$	if $P(n)$, then $Q(n)$
5	false	false	true
6	true	true	true
8	true	true	true

In fact, no matter what value you take n to be, implication (2) is always true. This is because it can never happen that n is an even integer but n^2 is not an even integer. So the following universal statement is true:

For all $n \in \mathbb{N}$, if n is an even integer, then n^2 is an even integer. (3)

(You will be asked to prove this statement formally later in the unit.)

By convention, we usually write statement (3) with the phrase 'for all $n \in \mathbb{N}$ ' omitted. In other words, usually when you see statement (2) written down, *it actually means* statement (3). So you would usually think of statement (2) as a true statement.

The general convention is as follows.

Convention for implications

The universal statement

for all $x \in X$, if $P(x)$, then $Q(x)$

is usually written simply as

if $P(x)$, then $Q(x)$.

As another example of this convention, let's look again at the first implication involving the variable n that you met in this subsection:

If n is an even integer, then n is a multiple of 4. (4)

You saw that this implication is sometimes true and sometimes false, depending on the value of n . So the following universal statement is false:

For all $n \in \mathbb{N}$, if n is an even integer, then n is a multiple of 4. (5)

By the convention in the box above, usually if you see statement (4) written down you should interpret it as statement (5), so you should conclude that it is false.

We've been using the convention above in this unit up till now, and we'll continue to use it in the rest of the unit. In the next section of this unit, you'll learn how to show that many universal statements of this form are true.

To show that an implication of the form 'if $P(n)$, then $Q(n)$ ' is false, you just need to find *one* value of n for which $P(n)$ is true but $Q(n)$ is false. That is, you need to find a counter-example. (You met the idea of a counter-example in Subsection 1.3.)

Note that if you want to decide whether an implication 'if $P(n)$, then $Q(n)$ ' is true or false, then considering values of n for which $P(n)$ is false will not help you (as shown by the truth table on page 32). For example, if you want to show that the implication

if n is prime, then $2n - 1$ is prime

is false, then there is no need to consider values of n that are not prime.



Example 6 *Using a counter-example to show that an implication is false*

Show that the following implication is false, by giving a counter-example.

If n is prime, then $2n - 1$ is prime.

Solution

You can write the implication as

if $P(n)$, then $Q(n)$

where $P(n)$ is the statement

n is prime

and $Q(n)$ is the statement

$2n - 1$ is prime.

Remember that you interpret the implication as meaning ‘if $P(n)$, then $Q(n)$, for *every* value of n ’; so to show that it is false you just need to find one value of n for which $P(n)$ is true but $Q(n)$ is false. The smallest value of n for which $P(n)$ is true is $n = 2$.

If $n = 2$, then n is prime and $2n - 1 = 3$, which is prime. So 2 is not a counter-example.

The next smallest value of n for which $P(n)$ is true is $n = 3$.

If $n = 3$, then n is prime and $2n - 1 = 5$, which is prime. So 3 is not a counter-example.

The next smallest value of n for which $P(n)$ is true is $n = 5$.

If $n = 5$, then n is prime and $2n - 1 = 9$, which is not prime. So 5 is a counter-example and the implication is false.

We have shown that $P(5)$ is true but $Q(5)$ is false and so the statement ‘if $P(n)$, then $Q(n)$ ’ is false.

In the next activity you can practise showing that an implication is false by finding a counter-example. The first four parts are more straightforward than the implication in Example 6 and, for these parts, you should be able to spot a counter-example relatively easily. The last part requires a bit more thought.

Activity 19 *Using counter-examples to show that implications are false*

For each of the following implications, show that the implication is false by finding a counter-example.

- (a) If n is a prime number, then n is odd.
- (b) If $x \in \mathbb{R}$, then $x^2 \leq 0$.
- (c) If n is odd, then $2n$ is odd.
- (d) If $x > 0$, then $x + 1 > 2$.
- (e) If n is prime, then $3n - 4$ is prime.

Implications can be written in many different ways. In particular, the implication ‘if P , then Q ’ can be written as

P implies Q .

Also, the word ‘implies’ can be replaced by the symbol ‘ $\Rightarrow$ ’. So the shortest way of writing an implication is to write

$P \Rightarrow Q$.

All the various ways of writing implications can of course be used with variable propositions. For example, you can write the implication

if $x < 0$, then $x^2 > 0$

as

$x < 0$ implies $x^2 > 0$,

or alternatively as

$x < 0 \Rightarrow x^2 > 0$.

Remember though, that however such an implication is written, it is usually interpreted as a universal statement rather than as a variable proposition.

Table 3 lists some different ways of writing an implication, and illustrates them using the example ‘if $x < 0$, then $x^2 > 0$ ’. It includes the ways mentioned above, and some further ways.

Table 3 Ways of writing an implication

Ways of writing 'if P , then Q '	Ways of writing 'if $x < 0$, then $x^2 > 0$ '
P implies Q	$x < 0$ implies $x^2 > 0$
$P \Rightarrow Q$	$x < 0 \Rightarrow x^2 > 0$
Q whenever P	$x^2 > 0$ whenever $x < 0$ (or: $x^2 > 0$ for all $x < 0$)
Q follows from P	$x^2 > 0$ follows from $x < 0$
P is sufficient for Q	$x < 0$ is sufficient for $x^2 > 0$
Q is necessary for P	$x^2 > 0$ is necessary for $x < 0$
P only if Q	$x < 0$ only if $x^2 > 0$

You may find the last three lines of Table 3 harder to understand than the others. It might help to think of the phrase ' P is sufficient for Q ' as being short for ' P being true is sufficient for Q to be true', and the phrase ' Q is necessary for P ' as being short for ' Q is necessarily true if P is true'. You may well find the last line the hardest of all to understand. You can think of it as being short for ' P is true only if Q is true'.

Activity 20 *Rewriting implications*

Rewrite each of the following implications in the form ' $\dots \Rightarrow \dots$ '.

- (a) If n is odd, then n^2 is odd.
- (b) n is a multiple of 6 only if n is a multiple of 3.
- (c) $x + 1 > 1$ whenever $x > 0$.
- (d) $x \geq 2$ only if $x^2 \geq 4$.
- (e) Every multiple of 4 is an even number.

Converses

The **converse** of the implication ‘if P , then Q ’ is the implication ‘if Q , then P ’. In other words, the converse of the implication $P \Rightarrow Q$ is the implication $Q \Rightarrow P$. For example, the converse of the implication

$$x < 0 \Rightarrow x^2 > 0$$

is the implication

$$x^2 > 0 \Rightarrow x < 0.$$

Notice that the first of these two implications is true, but the second is false. It can also happen that an implication and its converse are both true, or both false. For example, you’ll see later in this unit that the following implication and its converse are both true:

If n is odd, then n^2 is odd.

So it’s important to remember the following fact.

If an implication is true, then its converse *may* or *may not* be true.

It’s also important that you don’t confuse the *converse of an implication* with the *negation of a statement*. For example, the implication

$$x < 0 \Rightarrow x^2 > 0$$

has converse

$$x^2 > 0 \Rightarrow x < 0$$

and negation

$$x < 0 \text{ does not imply } x^2 > 0.$$

When you are using implications to prove a result, it is important to think carefully about whether you need to show that an implication is true or that its converse is true – many errors in proofs arise when these are mixed up.

If you think about the real-life example that was mentioned earlier, then this may help you to appreciate the difference between an implication and its converse. The example was

if it is raining, then the children play indoors.

The converse of this is

if the children play indoors, then it is raining.

Even if the original implication is true, the converse could be false as there are lots of other reasons why the children might play indoors – for example, the weather might be too hot!

Example 7 *Writing the converse of an implication*

Write down the converse of the following implication.

Every integer that is a multiple of 4 is also a multiple of 8.

Solution

 It will be easier to write the converse if you first rewrite the implication in the form ‘if P , then Q ’ or $P \Rightarrow Q$. 

This implication can be written as

if n is a multiple of 4, then n is a multiple of 8.

 The converse of ‘if P , then Q ’ is ‘if Q , then P ’, so swap the statement ‘ n is a multiple of 4’ with the statement ‘ n is a multiple of 8’. 

The converse of this is

if n is a multiple of 8, then n is a multiple of 4.

This can also be written as:

every integer that is a multiple of 8 is also a multiple of 4.

Activity 21 *Writing the converses of implications*

Write down the converse of each of the following implications. In each case, state whether you think the original implication is true and whether you think the converse is true. You do not need to justify your answers.

- (a) If n is odd, then $2n$ is odd.
- (b) $x + 1 > 1$ whenever $x > 0$.
- (c) n is a multiple of 6 only if n is a multiple of 3.
- (d) $x \geq 2$ if $x^2 \geq 4$.
- (e) Every multiple of 4 is an even number.

Equivalences

You have seen that, if an implication is true, then its converse is not necessarily true. For example, the implication

$$x < 0 \Rightarrow x^2 > 0$$

is true, but its converse is false. For some of the examples that you have seen, however, both the implication and its converse are true. You will see later in this unit that this is the case for the implication

$$\text{if } n \text{ is odd, then } n^2 \text{ is odd.} \tag{6}$$

Implications with the property that both the implication and its converse are true are used frequently in proofs.

To assert that both implication (6) and its converse are true, you can write

$$\text{if } n \text{ is odd, then } n^2 \text{ is odd, and if } n^2 \text{ is odd, then } n \text{ is odd.}$$

This statement is quite lengthy though, and you can write it more concisely as

$$n \text{ is odd if and only if } n^2 \text{ is odd.}$$

Mathematicians sometimes abbreviate the phrase ‘if and only if’ to ‘iff’. So the statement above can be written even more concisely as

$$n \text{ is odd iff } n^2 \text{ is odd.}$$

Also, the words ‘if and only if’ can be replaced by the symbol ‘ $\Leftrightarrow$ ’, to give

$$n \text{ is odd } \Leftrightarrow n^2 \text{ is odd.}$$

This neatly combines the implication ‘ n is odd $\Rightarrow n^2$ is odd’ and its converse ‘ n^2 is odd $\Rightarrow n$ is odd’ into one statement.

In general, any statement of the form ‘ P if and only if Q ’ is called an **equivalence**. Table 4 lists some different ways of writing an equivalence, using the equivalence

$$n \text{ is odd if and only if } n^2 \text{ is odd}$$

as an example.

Table 4 Ways of writing an equivalence

Ways of writing ‘ P if and only if Q ’	Ways of writing ‘ n is odd if and only if n^2 is odd’
$P \Leftrightarrow Q$	$n \text{ is odd } \Leftrightarrow n^2 \text{ is odd}$
$P \text{ iff } Q$	$n \text{ is odd iff } n^2 \text{ is odd}$
P is equivalent to Q	$n \text{ is odd is equivalent to } n^2 \text{ is odd}$
P is necessary and sufficient for Q	$n \text{ is odd is necessary and sufficient for } n^2 \text{ to be odd}$

Earlier you saw that by convention an implication involving a variable is usually interpreted as a universal statement asserting that the implication is true for all values of the variable. A similar convention holds for equivalences, as follows.

Convention for equivalences

The equivalence

for all $x \in X$, $P(x)$ if and only if $Q(x)$

is usually written simply as

$P(x)$ if and only if $Q(x)$.

Thus, when we write down an equivalence of the form $P(x) \Leftrightarrow Q(x)$, we are usually asserting that the equivalence is true for all values of x .

So, to show that an equivalence $P(x) \Leftrightarrow Q(x)$ is true (for all values of x) you need to show that both $P(x) \Rightarrow Q(x)$ and $Q(x) \Rightarrow P(x)$ are true (for all values of x). If you can find a counter-example to show that one of these implications is false for at least one value of x , then you'll know that the equivalence $P(x) \Leftrightarrow Q(x)$ is false.

Activity 22 *Understanding equivalences*

Consider the following statements.

$P(x)$ means: $x > 3$

$Q(x)$ means: $x < -3$

$R(x)$ means: $x^2 > 9$

Determine which of the following statements are true.

- (a) $P(x) \Rightarrow R(x)$.
- (b) $R(x) \Rightarrow P(x)$.
- (c) $P(x) \Leftrightarrow R(x)$.
- (d) $Q(x) \Rightarrow R(x)$.
- (e) $R(x) \Rightarrow Q(x)$.
- (f) $Q(x) \Leftrightarrow R(x)$.
- (g) $(P(x) \text{ OR } Q(x)) \Rightarrow R(x)$.
- (h) $R(x) \Rightarrow (P(x) \text{ OR } Q(x))$.
- (i) $(P(x) \text{ OR } Q(x)) \Leftrightarrow R(x)$.

When you are writing your own proofs, it is important that you think carefully about whether you want to write down an implication or an equivalence. Many proofs are incorrect because of a careless use of symbols, with $\Rightarrow$ being written down instead of $\Leftrightarrow$ and vice versa. You will see examples of such proofs in Subsection 3.4.

3 Direct proofs

In this section you will learn how to prove statements by using a sequence of implications or equivalences. The techniques in this section lie at the heart of most proofs.

3.1 Deduction

In the previous section, you saw many examples of implications; that is, statements of the form ‘if P , then Q ’, where P and Q are also statements. You saw that there are many ways of writing implications and that the shortest way of writing an implication is to write $P \Rightarrow Q$. In this subsection you’ll see how you can use implications to make deductions about the truth of other statements.

To prove that a mathematical statement is true, you usually have to apply a particular type of basic deductive step one or more times. To illustrate the idea, consider the real-life example of an implication that you met earlier:

If it is raining, then the children play indoors.

Suppose you know that this implication is true. Now consider the following statement:

It is raining.

If this statement is also true, then you can deduce from the implication above that the following statement is true:

The children play indoors.

You can express this argument symbolically. Let P and Q denote statements as follows:

P means: It is raining.

Q means: The children play indoors.

In the argument above, you saw that, if you know that the statements P and $P \Rightarrow Q$ are both true, then you can deduce that the statement Q is also true.

This form of deduction is the most typical single step in mathematical proofs. You establish the truth of two mathematical statements P and $P \Rightarrow Q$, and then deduce the truth of the mathematical statement Q .

Key deductive step

If the statements P and $P \Rightarrow Q$ are both true, then the statement Q is also true.

**Example 8** *Using the key deductive step*

Determine whether the following attempt to use the key deductive step is valid.

Suppose you know that:

If Julia is a fish, then Julia can swim.

Julia can swim.

You deduce that:

Julia is a fish.

Solution

Identify statements that you can label as P and Q .

Let P be the statement

Julia is a fish.

and Q be the statement

Julia can swim.

Identify any statements that you know to be true.

The statements that you know to be true are $P \Rightarrow Q$ and Q .

Check whether the deduction is valid.

The argument attempts to deduce that P is true. This is not a valid deduction.

Activity 23 *Using the key deductive step*

Below are five attempts at using the key deductive step. Determine which of these deductions are valid.

(a) You are told that:

If the last bus has gone, then you must find a taxi.

The last bus has gone.

You deduce that:

You must find a taxi.

(b) You are told that:

If Spot is a dog, then Spot has four legs.

Spot has four legs.

You deduce that:

Spot is a dog.

(c) You are told that:

If Winston is a cat, then Winston likes fish.

Winston is a cat.

You deduce that:

Winston likes fish.

(d) You are told that:

If it is raining, then the children play indoors.

It is not raining.

You deduce that:

The children do not play indoors.

(e) You are told that:

All sheep are green.

Dolly is a sheep.

You deduce that:

Dolly is green.

The next activity gives you the opportunity to practise using the key deductive step with some mathematical statements.

Activity 24 *Using the key deductive step with mathematical statements*

Below are three attempts at using the key deductive step. Determine which of these deductions are valid.

(a) You know that:

If n is odd, then $n = 2k + 1$ for some integer k .

n is odd.

You conclude that:

$n = 2k + 1$ for some integer k .

(b) You know that:

If $x > 2$, then $x^2 > 4$.

$x \leq 2$.

You conclude that:

$x^2 \leq 4$.

(c) You know that:

If n is a multiple of 4, then n is even.

n is even.

You conclude that:

n is a multiple of 4.

3.2 Proving an implication

For some of the statements that you met earlier in this unit, you were able to see immediately whether they were true or false. For example, it is clear that the following statement is true:

If $x < 0$, then $x^2 > 0$.

However, often you may think that a statement is true, but need a careful argument to prove that this is indeed the case. For example, you have seen the following statement in this unit:

If n is odd, then n^2 is odd.

It seems very reasonable that this statement should be true, but so far in this unit you have not seen a proof of it. Here is how you can prove that it is true.

Proof that if n is odd, then n^2 is odd

Let n be an odd integer. Then

$$n = 2k + 1$$

for some integer k . Hence

$$n^2 = (2k + 1)^2$$

and so

$$n^2 = 4k^2 + 4k + 1.$$

Thus

$$n^2 = 2(2k^2 + 2k) + 1$$

and so n^2 is an odd integer, since $2k^2 + 2k$ is an integer.

You may wonder how you would think of writing such a proof yourself. The following discussion of the key steps of this proof should help you to learn how to do this.

The idea is to consider an arbitrary integer n , and show that the implication $P \Rightarrow Q$ is true, where P is the statement

n is odd

and Q is the statement

n^2 is odd.

The proof follows the general principle that to prove a statement of the form $P \Rightarrow Q$ is true you must show that Q is true whenever P is true. (When P is false; that is, when n is even, the implication $P \Rightarrow Q$ is automatically true, so no more needs saying about such cases.)

So, the proof begins by assuming that P is true; that is, it assumes that the arbitrary integer n is odd. This is followed by a sequence of deductions that allow you to deduce that Q is true. More precisely, the first step is to deduce that the statement P_1 is true, where P_1 is the statement

$n = 2k + 1$, for some integer k .

(You know that $P \Rightarrow P_1$ is true, so you can use the key deductive step to deduce that P_1 is true.)

The second step is to deduce that the statement P_2 is true, where P_2 is the statement

$$n^2 = (2k + 1)^2.$$

(You know that $P_1 \Rightarrow P_2$ is true, so you can use the key deductive step to deduce that P_2 is true.)

The third step is to deduce that the statement P_3 is true, where P_3 is the statement

$$n^2 = 4k^2 + 4k + 1.$$

(You know that $P_2 \Rightarrow P_3$ is true, so you can use the key deductive step to deduce that P_3 is true.)

The fourth step is to deduce that the statement P_4 is true, where P_4 is the statement

$$n^2 = 2(2k^2 + 2k) + 1.$$

(You know that $P_3 \Rightarrow P_4$ is true, so you can use the key deductive step to deduce that P_4 is true.)

The final step is to deduce that Q is true. (You know that $P_4 \Rightarrow Q$ is true, so you can use the key deductive step to deduce that Q is true.)

This completes the proof that $P \Rightarrow Q$ is true for an arbitrary integer n , and hence for all $n \in \mathbb{N}$. Notice, however, that this final remark is left unsaid in the proof. Proofs usually do not include final remarks of this sort.

This description of the proof process is a particular example of the general strategy given below, which you can use to construct proofs of many different implications.

Strategy:

To prove an implication $P \Rightarrow Q$ using a sequence of deductions

1. Assume that P is true.
2. Identify a statement P_1 for which you know that $P \Rightarrow P_1$ is true, and deduce that P_1 is true.
3. Identify a statement P_2 for which you know that $P_1 \Rightarrow P_2$ is true, and deduce that P_2 is true.
4. Continue this process until you obtain a statement P_n for which you know that $P_n \Rightarrow Q$ is true.
5. Deduce that Q is true.

When using this strategy, you may find that it is not obvious what statement you should choose for the next one in the sequence. Don't be afraid to try a particular statement and, if you then get stuck, go back and try a different one. The very best mathematicians produce proofs in this way – but you don't usually get to see their rough work! To try to avoid too many steps in unhelpful directions, always keep an eye on where you are trying to get to.

In this unit, you are asked to practise proving results by proving simple facts about integers. As in the proof that you saw earlier in this subsection, it often helps to use the following basic facts about odd and even integers.

Odd and even integers

An integer n is **odd** if and only if $n = 2k + 1$ for some integer k .

An integer n is **even** if and only if $n = 2k$ for some integer k .

Note that when you are writing out a proof, you don't need to refer to implications directly. Instead, you can use various words to express the fact that one statement can be deduced from another.



Example 9 Proving a statement about odd integers

Prove that the sum of two odd integers is even, by using a sequence of deductions.

Solution

Begin by interpreting the statement that you need to prove in the form of an implication. The statement is an implication of the form $P \Rightarrow Q$, where P is the statement

m and n are both odd

and Q is the statement

$m + n$ is even.

Follow the strategy and begin by assuming that P is true.

Let m and n be odd integers.

Now make a deduction that will help to show that Q is true.

Then

$$m = 2k + 1 \text{ and } n = 2l + 1 \text{ for some integers } k \text{ and } l.$$

The statement Q is about the sum of m and n , so we add these two expressions together.

Thus

$$m + n = 2k + 1 + 2l + 1 = 2k + 2l + 2.$$

You want to show that the sum is even, so look to see if you can deduce that the sum can be written in the form $2p$ for some integer p .

Hence

$$m + n = 2(k + l + 1)$$

You can now deduce that Q is true.

and so $m + n$ is even, since $k + l + 1$ is an integer.

When you're proving a result by using a sequence of deductions, it is very important that you check carefully that each deduction that you make is valid. Many false proofs contain an invalid deduction hidden in the middle!

For example, here is a false proof that $1 = 0$.

Activity 25 *Identifying an error in a proof*

Identify the error in the following false proof that $1 = 0$.

Let $x = 1$. Then

$$x^2 = x.$$

Subtracting 1 from both sides gives

$$x^2 - 1 = x - 1.$$

Factorising the left-hand side gives

$$(x + 1)(x - 1) = x - 1$$

and dividing through by $x - 1$ gives

$$x + 1 = 1.$$

Hence $x = 0$. The original assumption was that $x = 1$, so this shows that $1 = 0$.

This activity illustrates the need to check the validity of each deduction carefully. It also illustrates the way a brief justification of each deduction can help the reader follow a proof more easily and check for any errors. The amount of justification that you should give in a proof depends both on the complexity of the material and on who you expect to be reading the proof.

It is a good idea to look at any proof that you produce to see whether you can simplify it; you will often find that your first attempt at producing a proof of a statement is unnecessarily complicated. This happens to the very best mathematicians!

In the following activity you can practise proving statements by using a sequence of deductions. Remember that in the proof in Example 9 it was not always written that one statement in the sequence implied the next. Instead there was a mixture of phrases such as 'hence', 'thus', 'then' and 'and so'. It is usual to write proofs in this way and mathematicians often try to use a variety of such phrases to avoid repetition. You can use whichever phrase of this type seems appropriate.

Activity 26 *Proving statements about odd and even integers*

Prove that each of the following statements is true, by using a sequence of deductions.

- (a) If n is an even integer, then n^2 is an even integer.
- (b) The sum of an odd integer and an even integer is odd.

Here is another example to show how the strategy that you have met in this subsection can be used.

Example 10 *Proving a statement about divisibility*

Prove that the following statement is true, by using a sequence of deductions.

Every multiple of 6 is also a multiple of 3.

Solution

 Begin by rewriting the statement that you need to prove in the form of an implication. The statement is an implication of the form $P \Rightarrow Q$, where P is the statement

n is a multiple of 6

and Q is the statement

n is a multiple of 3.

Follow the strategy and begin by assuming that P is true. 

Let n be a multiple of 6.

 You now need to make a deduction that will help to show that Q is true. 

Then

$$n = 6k \text{ for some integer } k.$$

 The statement Q is about the divisibility of n by 3, so look to see if the expression above has a factor of 3. 

Thus

$$n = 3(2k) \text{ for some integer } k$$

 You can now deduce that Q is true. 

and so n is a multiple of 3.

Activity 27 *Proving a statement about divisibility*

Prove that the following statement is true, by using a sequence of deductions.

Every multiple of 4 is even.

The next example shows how the strategy can be used to prove a result about a function.

Example 11 *Proving a statement about a function*

Let $f(x) = 2x + 1$. Prove that f is one-to-one, by using a sequence of deductions.

Solution

According to the definition of a one-to-one function, you need to show that, for all real numbers x_1 and x_2 , $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, or equivalently, that

if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

This is an implication of the form $P \Rightarrow Q$, with P being the statement

$$f(x_1) = f(x_2)$$

and Q being the statement

$$x_1 = x_2.$$

Follow the strategy and begin by assuming that P is true.

Let x_1 and x_2 be real numbers and suppose that $f(x_1) = f(x_2)$.

You now need to make a deduction that will help to show that Q is true. By using the formula for f you can deduce a relationship between x_1 and x_2 .

Then

$$2x_1 + 1 = 2x_2 + 1$$

Simplify this relationship as much as possible.

and hence

$$2x_1 = 2x_2.$$

You can now deduce that Q is true and hence that f is one-to-one.

Thus $x_1 = x_2$ and so f is one-to-one.

Activity 28 *Proving a statement about a function*

Show that the following statement is true, by using a sequence of deductions.

The function $f(x) = 4x + 3$ is one-to-one.

To end this subsection, here are two things that it's useful for you to appreciate.

You've seen that to prove an implication $P \Rightarrow Q$, the standard strategy is to assume that P is true, then deduce another statement P_1 , then another statement P_2 , and so on, until eventually you deduce Q .

If the deductions in such a proof are particularly straightforward, then you can sometimes present the proof in a slightly different form. Instead of starting by *assuming* P , and then eventually *deducing* Q , you can just present the sequence of implications

$$P \Rightarrow P_1, \quad P_1 \Rightarrow P_2, \quad \dots, \quad P_n \Rightarrow Q,$$

which you can write more concisely as

$$P \Rightarrow P_1 \Rightarrow P_2 \Rightarrow \dots \Rightarrow P_n \Rightarrow Q.$$

It follows directly from this sequence of implications that $P \Rightarrow Q$.

For example, consider the following proof, which is written in the usual style that you've met in this subsection.

Proof that if $x > 3$, then $(x - 1)^2 > 4$

Assume that $x > 3$. Then

$$x - 1 > 2$$

and so

$$(x - 1)^2 > 2^2.$$

It follows that

$$(x - 1)^2 > 4.$$

You can write this proof as a sequence of implications as follows.

Proof that if $x > 3$, then $(x - 1)^2 > 4$

$$\begin{aligned}x &> 3 \\ \Rightarrow x - 1 &> 2 \\ \Rightarrow (x - 1)^2 &> 2^2 \\ \Rightarrow (x - 1)^2 &> 4.\end{aligned}$$

This alternative style of proof of an implication is suitable only in very straightforward cases, such as those that involve the manipulation of an equation or an inequality. However it is often useful in such cases.

The second thing to appreciate is that the process of deducing each new statement in a proof from the one before often makes use of additional known facts, results or definitions. For example, the deductions in the box above make use of the fact that if you add a number to both sides of a true inequality, or square both sides of a true inequality if you know that both sides are positive, then you obtain another true inequality. In some other proofs, you've seen how deductions can make use of the definitions of odd and even integers. Sometimes, deductions make use of previously proved theorems. For example, many geometric proofs involve deductions that make use of Pythagoras' theorem.

3.3 Proving an equivalence

You have just seen how to use a sequence of deductions to prove that an implication $P \Rightarrow Q$ is true. The key idea was to assume that the statement P is true and then make a sequence of valid deductions until you reached the statement Q . This enabled you to conclude that the statement $P \Rightarrow Q$ is true. In this subsection you will see how this idea can be extended to show that an equivalence $P \Leftrightarrow Q$ is true.

Remember that an equivalence $P \Leftrightarrow Q$ is, by definition, the statement

$$P \Rightarrow Q \text{ and } Q \Rightarrow P.$$

So, to prove that such an equivalence is true, you have to show not only that the statement $P \Rightarrow Q$ is true, but also that its converse $Q \Rightarrow P$ is true. You may be able to do this by first using a sequence of deductions to prove that $P \Rightarrow Q$ is true, and then using another sequence of deductions to prove that $Q \Rightarrow P$ is true. This is illustrated in the next example.

**Example 12** *Proving an equivalence*

Prove that the following statement is true.

$n - 3$ is odd if and only if $3n - 2$ is even.

Solution

The statement is an equivalence $P \Leftrightarrow Q$, where P is the statement

$n - 3$ is odd

and Q is the statement

$3n - 2$ is even.

First assume that P is true and try to show that Q is true.

Let n be an integer such that $n - 3$ is odd. Then

$$n - 3 = 2k + 1$$

for some integer k . Hence

$$n = 2k + 4.$$

So

$$\begin{aligned} 3n - 2 &= 3(2k + 4) - 2 \\ &= 6k + 10 \\ &= 2(3k + 5). \end{aligned}$$

Thus $3n - 2$ is even, since $3k + 5$ is an integer.

Now assume that Q is true and try to show that P is true.

Let n be an integer such that $3n - 2$ is even. Then

$$3n - 2 = 2k$$

for some integer k . Subtracting $2n + 1$ from both sides gives

$$\begin{aligned} n - 3 &= 2k - 2n - 1 \\ &= 2(k - n - 1) + 1. \end{aligned}$$

So $n - 3$ is odd, since $k - n - 1$ is an integer.

The next activity asks you to prove a similar equivalence.

Activity 29 *Proving an equivalence*

Prove that the following statement is true.

$n + 2$ is odd if and only if $5n - 1$ is even.

In Example 12 and Activity 29 you saw how to prove a statement of the form $P \Leftrightarrow Q$ by finding one sequence of deductions to show that $P \Rightarrow Q$ is true, and another, quite separate, sequence of deductions to show that $Q \Rightarrow P$ is true. Sometimes, however, it is possible to find a single sequence of statements $P, P_1, P_2, \dots, P_n, Q$, starting with P and ending with Q , such that each statement is *equivalent* to its predecessor in the sequence. That is, the following sequence of equivalences holds:

$$P \Leftrightarrow P_1, \quad P_1 \Leftrightarrow P_2, \quad \dots, \quad P_n \Leftrightarrow Q.$$

If you can find such a sequence of equivalences, then you have obtained both sequences of deductions,

$$P \Rightarrow P_1, \quad P_1 \Rightarrow P_2, \quad \dots, \quad P_n \Rightarrow Q$$

and

$$Q \Rightarrow P_n, \quad \dots, \quad P_2 \Rightarrow P_1, \quad P_1 \Rightarrow P,$$

at the same time.

Actually, you will have seen this process in action many times when rearranging equations. The next example should remind you how it works. In this example, a sequence of equivalences of the form above is written concisely in the form

$$P \Leftrightarrow P_1 \Leftrightarrow P_2 \Leftrightarrow \dots \Leftrightarrow P_n \Leftrightarrow Q.$$

It is often convenient to do this when each equivalence in a sequence of equivalences is simple and clearly true.

Example 13 *Proving that two equations are equivalent*

Prove that the following statement is true, by using a sequence of equivalences.

$$y = 3x + 4 \text{ if and only if } x = \frac{1}{3}(y - 4).$$

Solution

 Start by writing down one of the equations. 

$$y = 3x + 4$$

 We want to make x the subject of the formula, so start to isolate it by subtracting 4 from both sides. This gives an equivalent equation. 

$$\Leftrightarrow y - 4 = 3x$$

 Dividing through by 3 and swapping sides gives another equivalent equation. 

$$\Leftrightarrow x = \frac{1}{3}(y - 4).$$

In this simple example the rules for rearranging equations ensure that the equivalence between each pair of equations holds. In more complicated examples great care is needed to check the validity of each step. Even when rearranging equations you need to be wary of various pitfalls. For example, if the two sides of an equation can take opposite signs, then squaring both sides of the equation will not give an equivalent equation, as this step cannot be reversed by taking the square root of each side. Also, division by zero is something you should always guard against.

Activity 30 *Proving that two equations are equivalent*

Prove that the following statement is true, by using a sequence of equivalences.

$$2x + 3y = 6 \text{ if and only if } y = 2 - \frac{2}{3}x.$$

You have now seen two methods for proving an equivalence. These are both summarised in the following box.

Strategy:**To prove an equivalence $P \Leftrightarrow Q$**

EITHER:

Use the strategy in Subsection 3.2 to show that both

$$P \Rightarrow Q \quad \text{and} \quad Q \Rightarrow P$$

are true.

OR:

1. Write down P .
2. Identify a statement P_1 for which you know that $P \Leftrightarrow P_1$ is true.
3. Identify a statement P_2 for which you know that $P_1 \Leftrightarrow P_2$ is true.
4. Continue this process until you obtain a statement P_n for which you know that $P_n \Leftrightarrow Q$ is true.
5. Deduce that $P \Leftrightarrow Q$ is true.

3.4 Proving a statement that is neither an implication nor an equivalence

So far we have concentrated on proving implications and equivalences. In this subsection we will look at how to prove a statement P that is neither of these.

Suppose that you want to prove such a statement P . One approach is to start with some statement Q that you know to be true and that appears to be relevant, and try to find a sequence of deductions

$$Q \Rightarrow P_1, \quad P_1 \Rightarrow P_2, \quad \dots, \quad P_n \Rightarrow P.$$

You can then deduce that P is true.

This approach often works, but it can require a lot of ingenuity to select the appropriate statement Q to begin with. Because of this, it is tempting to try a different approach. You may be tempted to think that you can prove a statement P by assuming that P is true and making a sequence of valid deductions until you obtain a statement Q that you know to be true. However, this does *not* imply that P is true, as this would be an incorrect use of the key deductive step. For example, here is an incorrect proof that $1 = -1$ based on this type of reasoning.

Incorrect proof that $1 = -1$

Suppose that

$$1 = -1.$$

Squaring both sides of this equation gives

$$1^2 = (-1)^2.$$

We can then deduce that

$$1 = 1,$$

which is clearly true. Therefore the original statement that $1 = -1$ must be true.

This argument is obviously incorrect! In other cases, you may find it much harder to spot that an incorrect argument has been used.

For example, here is an incorrect proof of another result.

Incorrect proof that $x^2 + y^2 \geq 2xy$, for all real numbers x and y

Let x and y be real numbers and suppose that

$$x^2 + y^2 \geq 2xy.$$

Then

$$x^2 + y^2 - 2xy \geq 0$$

and so

$$(x - y)^2 \geq 0.$$

The last inequality is clearly true, since the square of every real value is greater than or equal to 0. Therefore the original statement is true.

You are probably much more likely to believe this argument than the proof that $-1 = 1$, which is clearly incorrect. However, both proofs attempt to show that a statement is true by starting with the statement and making a valid sequence of deductions until a statement is obtained that is known to be true. This is probably the most common type of mistake that appears in proofs.

In fact, it *is* true that $x^2 + y^2 \geq 2xy$, for all real numbers x and y . The problem with the argument in the box above is that the implications are in the wrong direction. To produce a valid proof, you must begin with something that you know to be true and make a sequence of deductions that lead to the statement that you want to prove. Here is a correct proof of the result.

Correct proof that $x^2 + y^2 \geq 2xy$, for all real numbers x and y

Let x and y be real numbers. Then

$$(x - y)^2 \geq 0.$$

Thus

$$x^2 + y^2 - 2xy \geq 0$$

and hence

$$x^2 + y^2 \geq 2xy.$$

We now have a correct proof that $x^2 + y^2 \geq 2xy$, for all real numbers x and y . It is not, however, obvious how you would think of such a proof. You may have noticed that this argument looks exactly the same as the earlier incorrect argument but, at each stage of the argument, the converse of the original implication has been used. So, in the original incorrect argument, every deduction that we made could have been replaced by an observation that two statements are equivalent, as in the box below.

Alternative correct proof that $x^2 + y^2 \geq 2xy$, for all real numbers x and y

Let x and y be real numbers. Consider the statement

$$x^2 + y^2 \geq 2xy.$$

This is true if and only if

$$x^2 + y^2 - 2xy \geq 0,$$

which is true if and only if

$$(x - y)^2 \geq 0.$$

This last statement is true. Since we have just shown that it is equivalent to the statement $x^2 + y^2 \geq 2xy$, this statement must also be true.

This is now a correct proof, and you should find it easier to imagine constructing this proof than the earlier one. You can write the proof more briefly by using the $\Leftrightarrow$ symbol, as follows.

Concise correct proof that $x^2 + y^2 \geq 2xy$, for all real numbers x and y

Let x and y be real numbers. Then

$$\begin{aligned} x^2 + y^2 &\geq 2xy \\ \Leftrightarrow x^2 + y^2 - 2xy &\geq 0 \\ \Leftrightarrow (x - y)^2 &\geq 0. \end{aligned}$$

This last inequality is clearly true and so it must also be true that

$$x^2 + y^2 \geq 2xy.$$

This proof makes repeated use of the following fact.

If the statements Q and $P \Leftrightarrow Q$ are both true, then the statement P is also true.

This step forms the basis for the following strategy.

Strategy:

To prove a statement P using a sequence of equivalences

1. Write down P .
2. Identify a statement P_1 for which you know that $P \Leftrightarrow P_1$ is true.
3. Identify a statement P_2 for which you know that $P_1 \Leftrightarrow P_2$ is true.
4. Continue this process until you obtain a statement P_n for which you know that $P_n \Leftrightarrow Q$ is true and Q is true.
5. Deduce that P is true.

The next example shows you how you can use this strategy.



Example 14 *Proving an inequality*

Prove that

$$(2n + 1)^2 \geq 4n + 5$$

for every natural number n , by using a sequence of equivalences.

Solution

Follow the strategy and write down the statement that you are trying to prove.

Let n be a natural number. Then

$$(2n + 1)^2 \geq 4n + 5$$

Write down a statement that is equivalent to the one that you have just written. The simplest way to do this is to expand the square on the left-hand side.

$$\Leftrightarrow 4n^2 + 4n + 1 \geq 4n + 5$$

Now write down an equivalent statement by subtracting $4n + 1$ from both sides in order to simplify the expressions.

$$\Leftrightarrow 4n^2 \geq 4$$

Now write down a simpler equivalent statement by dividing by 4.

$$\Leftrightarrow n^2 \geq 1.$$

This last statement is clearly true for every natural number n .

This last inequality is true and so the original inequality is also true.

Activity 31 *Proving inequalities*

Prove that each of the following inequalities is true, by using a sequence of equivalences.

- (a) $x^2 \geq (x + 2)(x - 2)$, for all real x .
 (b) $(2n + 2)(5n + 1) > (3n + 2)^2$, for all $n \geq 2$.

4 Proof by induction

In this section you will learn how to use a method of proof known as *mathematical induction*. This method builds on the idea that you met in the previous section of proving a statement by using a sequence of deductions. It is useful for proving that a variable proposition $P(n)$ is true for *every* natural number n .

4.1 Mathematical induction

Consider the following statement that you saw at the beginning of this unit:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2, \text{ for all } n \in \mathbb{N}.$$

If you use $P(n)$ to denote the statement

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2,$$

then you can easily check that $P(n)$ is true for small values of n , as you did in Activity 10. For example,

$$\begin{aligned} 1 &= 1^2, \\ 1 + 3 &= 4 = 2^2, \\ 1 + 3 + 5 &= 9 = 3^2, \end{aligned}$$

so certainly $P(1)$, $P(2)$ and $P(3)$ are all true.

Later in this subsection, you will use **mathematical induction** to prove that $P(n)$ is true *for all* positive integers n . The basic idea of the proof is to show that the implication

$$P(k) \Rightarrow P(k + 1)$$

is true for $k = 1, 2, \dots$. Then, because you know that $P(1)$ is true, you can deduce from the key deductive step that $P(2)$ is true. Similarly, you can then deduce that $P(3)$ is true, and then that $P(4)$ is true, and so on. It follows that you can show that $P(n)$ is true for each $n \in \mathbb{N}$.

The logical structure of this method of proof is as follows.

Strategy:

To prove by mathematical induction that a variable proposition $P(n)$ is true for all $n \in \mathbb{N}$

1. Show that $P(1)$ is true.
2. Show that the implication $P(k) \Rightarrow P(k + 1)$ is true for $k = 1, 2, \dots$
3. Deduce that $P(n)$ is true for all $n \in \mathbb{N}$.

Mathematical induction enables you to deduce the truth of many variable propositions $P(n)$ for *all* natural numbers n . The power of mathematical induction lies in the fact that it is often easier to establish the truth of the statement

$$P(k) \Rightarrow P(k + 1)$$

than it is to establish the truth of $P(n)$ directly.

It is important to remember that any proof by mathematical induction contains two elements. The second or **inductive** step is to show that, if $P(k)$ is true, then $P(k + 1)$ is also true. It is very important that you also

remember to carry out the first step and check that $P(1)$ is true so that the chain of deductions can start.

As a visualisation of the method of mathematical induction, consider an unending line of dominoes, set up so that:

if any domino falls to the right, then it will knock over the next domino to the right.

(This represents the inductive step $P(k) \Rightarrow P(k+1)$.) If none of the dominoes is disturbed, then they will all remain standing; see Figure 5(a). But suppose that the first domino is knocked down; see Figure 5(b) – this represents the first step $P(1)$ being true. Then the first domino will knock over the second; see Figure 5(c). The second will knock over the third, the third will knock over the fourth, and so on. Thus all the dominoes are knocked down.



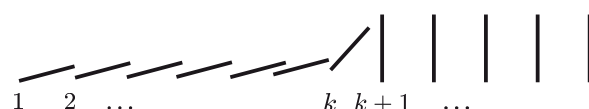
(a)



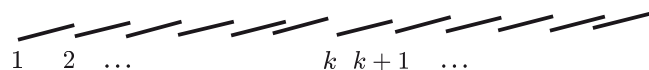
(b)



(c)



(d)



(e)

Figure 5 A visualisation of mathematical induction using dominoes

The next example shows how mathematical induction can be used to prove a formula for the sum of the squares of the first n natural numbers. (You saw this formula in MST124 Unit 10, but no proof was given there.)



Example 15 Proving a formula for a sum

Prove that the following statement is true, by using mathematical induction.

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1), \text{ for all } n \in \mathbb{N}.$$

Solution

Specify a statement $P(n)$.

Let $P(n)$ be the statement

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

Show that $P(1)$ is true.

Then $P(1)$ is true, because

$$1^2 = 1 \quad \text{and} \quad \frac{1}{6}(1+1)(2+1) = 1.$$

Show that the implication $P(k) \Rightarrow P(k+1)$ is true for $k = 1, 2, \dots$

Now let $k \geq 1$, and assume that $P(k)$ is true; that is,

$$1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1).$$

We want to deduce that $P(k+1)$ is true; that is,

$$1^2 + 2^2 + \cdots + (k+1)^2 = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1).$$

Write down the left-hand side of this equation, and aim to show that it's equal to the right-hand side. Start by using the fact that $P(k)$ is true.

Now

$$\begin{aligned} 1^2 + 2^2 + \cdots + (k+1)^2 &= (1^2 + 2^2 + \cdots + k^2) + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \quad (\text{by } P(k)) \end{aligned}$$

Now try to rearrange this expression into the expression that you're aiming for. Start by taking out the factor $\frac{1}{6}(k+1)$. That looks helpful since the expression that you're aiming for has $\frac{1}{6}(k+1)$ as a factor.

$$\begin{aligned} &= \frac{1}{6}(k+1)((2k^2+k)+6(k+1)) \\ &= \frac{1}{6}(k+1)(2k^2+7k+6) \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1). \end{aligned}$$

 This is the expression that you're aiming for. 

Thus $P(k+1)$ is true if $P(k)$ is true; that is,

$$P(k) \Rightarrow P(k+1), \text{ for } k = 1, 2, \dots$$

Hence, by mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

In the following activity you can try using proof by induction yourself.

Activity 32 Proving formulas for sums

Prove that the following statements are true, by using mathematical induction.

- (a) $1 + 3 + 5 + \dots + (2n - 1) = n^2$, for all $n \in \mathbb{N}$.
- (b) $1^3 + 2^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2$, for all $n \in \mathbb{N}$.

The process of mathematical induction can be used in a variety of different contexts. You will now see how it can be used to establish a formula for the n th derivative of a particular function. Suppose that you want an expression for the n th derivative of the function

$$f(x) = \frac{1}{1-x}.$$

If you calculate the first few derivatives of this function then you will see that

$$f'(x) = \frac{1}{(1-x)^2},$$

$$f''(x) = \frac{2}{(1-x)^3},$$

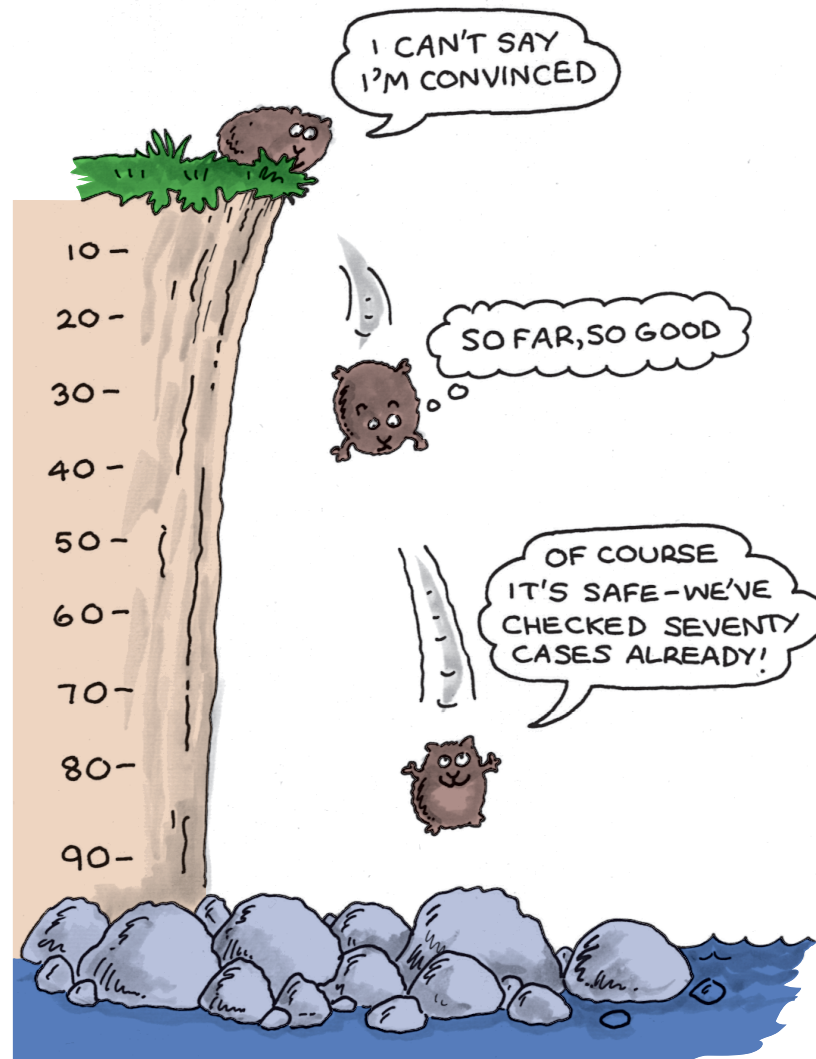
$$f'''(x) = \frac{3 \times 2}{(1-x)^4} = \frac{6}{(1-x)^4}.$$

You might observe that these derivatives show a pattern that can be generalised by the formula

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \quad \text{for all } n \in \mathbb{N}. \quad (7)$$

This approach of 'specialise/generalise' is a very useful way of discovering solutions to problems. You do, however, need to be careful; the fact that

the first few cases of some general statement are true does not guarantee that the statement is true in general.



‘Generalisation requires proof’

The next example shows how mathematical induction can be used to prove that equation (7) is true for all natural numbers n .


Example 16 *Proving a formula for an n th derivative*

Let

$$f(x) = \frac{1}{1-x}.$$

Show that

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \quad \text{for all } n \in \mathbb{N},$$

by using mathematical induction.

Solution

Specify a statement $P(n)$.

Let $P(n)$ be the statement

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}.$$

Show that $P(1)$ is true.

Then $P(1)$ is true, because, by the chain rule,

$$f'(x) = \frac{-1}{(1-x)^2} \times (-1) = \frac{1}{(1-x)^2}.$$

Show that the implication $P(k) \Rightarrow P(k+1)$ is true for $k = 1, 2, \dots$

Now let $k \geq 1$, and assume that $P(k)$ is true; that is,

$$f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}.$$

We want to deduce that $P(k+1)$ is true; that is,

$$f^{(k+1)}(x) = \frac{(k+1)!}{(1-x)^{k+2}}.$$

To find the $(k+1)$ th derivative of f , differentiate $f^{(k)}(x)$. This gives

$$\begin{aligned}
f^{(k+1)}(x) &= \frac{d}{dx} \left(f^{(k)}(x) \right) \\
&= \frac{d}{dx} \left(\frac{k!}{(1-x)^{k+1}} \right) \quad (\text{by } P(k)) \\
&= \frac{-(k+1) \times k!}{(1-x)^{k+2}} \times (-1) \quad (\text{by the chain rule}) \\
&= \frac{(k+1)!}{(1-x)^{k+2}}.
\end{aligned}$$

Thus

$$P(k) \Rightarrow P(k+1), \text{ for } k = 1, 2, \dots$$

Hence, by mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Activity 33 Proving a formula for an n th derivative

Let

$$f(x) = xe^x.$$

Show that

$$f^{(n)}(x) = (n+x)e^x, \quad \text{for all } n \in \mathbb{N},$$

by using mathematical induction.

4.2 Mathematical induction starting at a number other than 1

In this subsection you'll see how to use a generalisation of the method of mathematical induction. To illustrate the idea, let $P(n)$ be the following variable proposition, which is an inequality:

$$2^n > 4n.$$

This inequality is not true for all natural numbers n . For example, when $n = 1$,

$$2^n = 2^1 = 2 \text{ and } 4n = 4 \times 1 = 4,$$

so $P(1)$ is false.

Activity 34 *Checking whether a variable proposition is true or false*

Let $P(n)$ be the statement

$$2^n > 4n.$$

You saw above that $P(1)$ is false. Determine whether $P(n)$ is true or false for each of $n = 2, 3, 4, 5, 6$.

You saw in the activity above that, for the variable proposition $P(n)$ stated there, the statements $P(1)$, $P(2)$, $P(3)$ and $P(4)$ are all false. However, for larger values of n , the inequality does seem to hold, and with increasing ease.

You might wonder whether it is possible to use mathematical induction to prove that $P(n)$ is true for $n \geq 5$. Since you know that $P(5)$ is true, if you could show that the implication $P(k) \Rightarrow P(k+1)$ is true for $k \geq 5$ then you would be able to deduce that $P(6)$ is true and then that $P(7)$ is true, and so on.

In general, you can use *any* natural number, say N , for the initial step in mathematical induction.

Strategy:

To prove by mathematical induction that a variable proposition $P(n)$ is true for $n \geq N$

1. Show that $P(N)$ is true.
2. Show that the implication $P(k) \Rightarrow P(k+1)$ is true for $k \geq N$.
3. Deduce that $P(n)$ is true for $n \geq N$.

Again, it may help you to visualise this method using dominoes. Suppose as before that each domino (at least from the N th onwards) has been set up close enough to the next that you can be sure of the following:

if any domino falls to the right, then it will knock over the next domino to the right.

If none of them is disturbed, then the dominoes will remain standing; see Figure 6(a). But suppose that one domino is knocked down, this time the N th; see Figure 6(b). Then it will knock over the $(N+1)$ th; see Figure 6(c). That will knock over the next, and so on.

Thus all the dominoes to the right of the N th are knocked down.

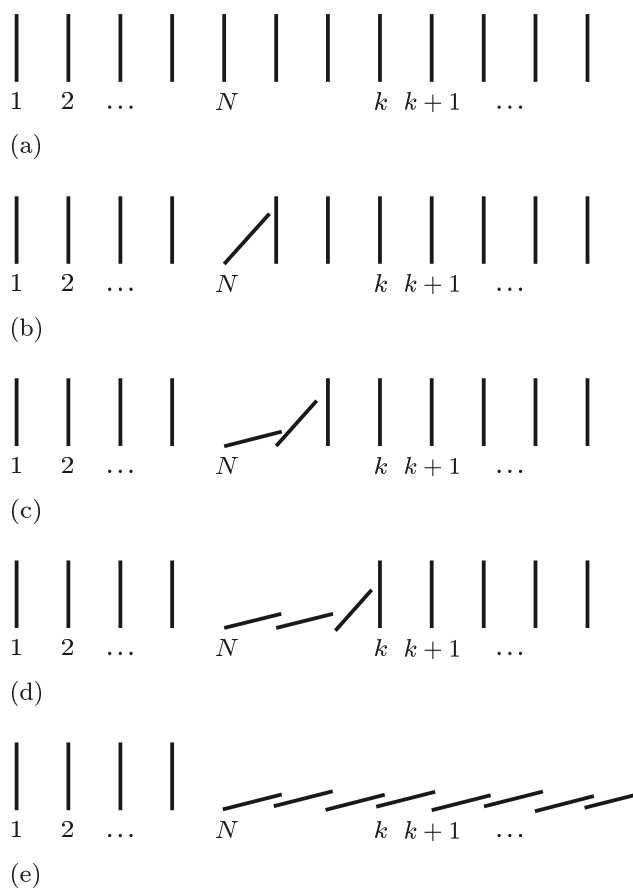


Figure 6 A visualisation of generalised mathematical induction using dominoes: the dominoes from the N th onwards are knocked down

In the next example you will see how to use this generalised method of induction to show that the inequality that you looked at earlier does indeed hold for all $n \geq 5$.



Example 17 Proving an inequality

Prove that $2^n > 4n$ for $n \geq 5$, by using mathematical induction.

Solution

Specify a statement $P(n)$.

Let $P(n)$ be the statement

$$2^n > 4n.$$

 Show that $P(5)$ is true. 

Then $P(5)$ is true, because

$$2^5 = 32 > 20 = 4 \times 5.$$

 Show that the implication $P(k) \Rightarrow P(k+1)$ is true for $k \geq 5$. 

Now let $k \geq 5$, and assume that $P(k)$ is true; that is, $2^k > 4k$. We want to deduce that $P(k+1)$ is true; that is, $2^{k+1} > 4(k+1)$. Now

$$\begin{aligned} 2^{k+1} &= 2 \times 2^k \\ &> 2 \times 4k \quad (\text{by } P(k)) \\ &= 8k. \end{aligned}$$

We would like to show that $8k \geq 4(k+1)$. Now

$$\begin{aligned} 8k &\geq 4(k+1) \\ \Leftrightarrow 8k &\geq 4k+4 \\ \Leftrightarrow 4k &\geq 4. \end{aligned}$$

This last inequality is clearly true since $k \geq 5$. Hence $8k \geq 4(k+1)$. Since we have already shown that $2^{k+1} > 8k$, it follows that $2^{k+1} > 4(k+1)$, as required. Thus

$$P(k) \Rightarrow P(k+1), \text{ for } k \geq 5.$$

Hence, by mathematical induction, $P(n)$ is true, for $n \geq 5$.

Part of the argument in Example 17 uses the fact that certain inequalities are equivalent to other inequalities. This part of the argument would not have been correct if we had just written that each inequality implied the next inequality. This is a good illustration of the type of argument where it is very important to make sure that you are using implication and equivalence signs correctly.

Activity 35 Proving inequalities

Prove that each of the following inequalities is true, by using mathematical induction.

- (a) $3^n > 10n$ for $n \geq 4$.
- (b) $n! > 3^n$ for $n \geq 7$.

5 Indirect proofs

In the previous sections you saw how to prove results directly by constructing a sequence of deductions. In this section you will meet two indirect methods of proving a result. With both methods you use a sequence of deductions to prove a related result, which enables you to establish the truth of the original result. At the end of the section you will see how one of these techniques can be used to prove Fermat's little theorem, which you met in your study of number theory in Unit 3.

5.1 Proof by contradiction

If you are trying to show that a statement is true, then it can sometimes help to think about what the consequences might be if the statement were false. For example, consider the following statement:

There are no integers m and n with $2m + 4n = 163$.

If we assume that this statement is false, then there exist integers m and n such that

$$2m + 4n = 163.$$

You can write this equation as

$$2(m + 2n) = 163.$$

The left-hand side of this equation is even, and the right-hand side is odd. This is impossible and so our assumption that the statement is false must have been incorrect. In other words, we have shown that the statement is true.

The method of proof used in this example is called **proof by contradiction**.

One of the most famous proofs by contradiction proves one of the statements that you saw at the beginning of the unit, namely that there are infinitely many primes.



G.H. Hardy (1877–1947)

The proof that there are infinitely many primes was first given by Euclid in about 300 BC. It was a favourite proof of the mathematician G.H. Hardy, who, in his fascinating book *A Mathematician's Apology*, described proof by contradiction as 'one of a mathematician's finest weapons'.

Here is the proof.

Proof that there are infinitely many primes

Assume that the statement is not true; in other words, assume that there are only finitely many primes. We want to deduce a contradiction.

To do this, denote the finitely many primes by $p_1, p_2, \dots, p_n$, and let

$$N = p_1 p_2 \cdots p_n + 1.$$

This integer is greater than each of the primes $p_1, \dots, p_n$, so it is not a prime (we've assumed that $p_1, \dots, p_n$ are the only primes). It must, therefore, have a prime factor, p say. Since p is prime, it is equal to one of the integers $p_1, \dots, p_n$, and must therefore leave the remainder 1 when divided into N . This, however, is a contradiction because p is a (prime) factor of N .

Thus our assumption that there are only finitely many primes is false and hence there are in fact infinitely many primes, as claimed.

The general idea of the method of proof by contradiction is as follows. Suppose that you want to show that a statement P is true. You begin the proof by assuming that the statement P is in fact false (or, in other words, you suppose that the negation of P is true). You then make a sequence of deductions that follow from this assumption until you reach a conclusion that you know to be false. We say that this conclusion is a **contradiction**. It shows that some part of the argument that you wrote down must be wrong. If all of the deductions that you made are correct, then the only part of the argument that could be wrong is your original assumption that P is false. This proves that P is in fact true. For clarity, the method is summarised in the following strategy.

Strategy:

To prove a statement P using proof by contradiction

1. Assume that P is false. In other words, assume that the negation Q of P is true.
2. Construct a sequence of statements $P_1, \dots, P_n$ such that all the implications

$$Q \Rightarrow P_1, \quad P_1 \Rightarrow P_2, \quad \dots, \quad P_{n-1} \Rightarrow P_n$$

are true, but P_n is false.

3. Since this is a contradiction, deduce that Q is false and hence that P is true.

The next example illustrates how to use this strategy.

**Example 18** *Using proof by contradiction*

Show that the following statement is true, by using proof by contradiction.

There is no natural number n such that $8n < 4(n + 1)$.

Solution

Begin by assuming that the statement that you want to prove is in fact false.

Suppose that the statement is not true; in other words, suppose that there exists a natural number n such that

$$8n < 4(n + 1).$$

Try to deduce a contradiction. Here you can do that by simplifying the inequality.

Then

$$8n < 4n + 4$$

and so

$$4n < 4.$$

Thus $n < 1$. This is a contradiction, since the smallest natural number is 1.

We can now deduce that the original statement is true.

Thus our assumption that the original statement was false is incorrect. This shows that the statement is true.

In the next activity you can practise using the strategy yourself.

Activity 36 *Using proof by contradiction*

Show that each of the following statements is true, by using proof by contradiction.

- (a) There are no integers m and n with $5m + 10n = 549$.
- (b) There is no real number x such that $x^2 + 1 < 2x$.
- (c) There is no integer n such that $n^2 < (n + 1)(n - 1)$.

In the next example you will see another classic proof by contradiction. It illustrates how this method can be used to show that certain numbers are irrational.

Example 19 *Using proof by contradiction to prove irrationality*

Show that $\sqrt{2}$ is irrational, by using proof by contradiction.

Solution

 Begin by assuming that the statement is false. 

Suppose that this statement is not true; in other words, suppose that $\sqrt{2}$ is rational.

 Try to deduce a contradiction. First try to use the definition of a rational number. 

Then there exist integers m and n such that $m/n = \sqrt{2}$.

 There are many possible choices of m and n ; choose values of m and n that have no common divisors. This is essential to make the proof work. 

Choose m and n such that they have no common divisors.

 Now square each side of the equation above and consider what this implies about the integers m and n . 

Then $m^2/n^2 = 2$; that is,

$$m^2 = 2n^2.$$

Hence m^2 is even. This implies that m is even (because if the prime 2 is a divisor of m^2 , then it must also be a divisor of m). Thus $m = 2k$ for some integer k . Substituting into the equation above gives

$$4k^2 = 2n^2$$

and hence

$$2k^2 = n^2.$$

In the same way as above, this implies that n is even. This, however, is a contradiction, since m and n have no common divisors.

 You can now deduce that the original statement is true. 

Thus the assumption that $\sqrt{2}$ is rational was false and hence $\sqrt{2}$ is irrational, as claimed.

When working through the proof above, you may have wondered how we knew to choose m and n such that they have no common divisors. Once you read further through the proof you will have seen why this condition is necessary. When proving a result yourself, you may need to go back to earlier parts of your proof and add in extra conditions once you have realised that they are necessary. You usually only get to see the tidied-up versions of other people's proofs and so don't get to see all the changes

that they have made as they've worked out what conditions are needed to make the proof work!

Activity 37 *Using proof by contradiction to prove irrationality*

Show that $\sqrt{5}$ is irrational, by using proof by contradiction.

5.2 Proof by contraposition

The final technique for proving results that you will meet in this unit is a technique for proving that an implication 'if P , then Q ' is true. The basic idea is to form a new implication that has the same meaning but is easier to prove.

To illustrate how you can rewrite an implication to form a new implication with the same meaning, consider the following real-life example of an implication:

If it is Friday today, then tomorrow is Saturday.

An equivalent way of stating this is as follows:

If tomorrow is not Saturday, then it is not Friday today.

You may want to convince yourself that these implications really do have the same meaning. Suppose that it is Friday today. The first implication tells you that it is Saturday tomorrow. Now think about what the second implication tells you, still supposing that it is Friday today. If you assume that tomorrow is not Saturday, then the second implication tells you that it is not Friday today. This is not true though, so your assumption that tomorrow is not Saturday must have been wrong. In other words, the second implication also tells you that it is Saturday tomorrow.

The original implication was written in terms of two simple statements:

it is Friday today

and

tomorrow is Saturday.

The second implication was written in terms of the negations of these statements:

it is not Friday today

and

tomorrow is not Saturday.

You met the negation of a statement in Subsection 2.1, where you saw that the negation of a statement P is often written as NOT P . Recall that NOT P is true if P is false, and false if P is true.

Any implication

if P , then Q

can be rewritten as

if NOT Q , then NOT P .

The second implication is known as the **contrapositive** of the first. So an implication and its contrapositive are equivalent.

When you are trying to form the contrapositive of a statement, you may find it helpful to refer back to Subsection 2.1, where you learned about the negation of a statement.

Activity 38 Forming contrapositives

Write down the contrapositive of each of the following statements.

- (a) If n^2 is even, then n is even.
- (b) If n^2 is odd, then n is odd.
- (c) If n is odd, then n is not a multiple of 4.
- (d) If $x > 0$, then $x + 1 > 1$.

To confirm that the contrapositive of an implication really is equivalent to the original implication, you may find it helpful to consider the truth tables for the two implications. You saw in Subsection 2.4 that the implication ‘if P , then Q ’ has the following truth table.

P	Q	if P , then Q
true	true	true
true	false	false
false	true	true
false	false	true

The truth table for the contrapositive of ‘if P , then Q ’ is as follows.

P	Q	NOT P	NOT Q	if NOT Q , then NOT P
true	true	false	false	true
true	false	false	true	false
false	true	true	false	true
false	false	true	true	true

Since the values in the final columns of both tables agree, the implication and its contrapositive are equivalent.

If you want to prove a particular implication, and you are finding it hard to use a direct method of proof, then you may find it easier to show that the contrapositive of the implication is true. This technique is known as

proof by contraposition. For example, suppose that you want to show that the following implication is true:

if n^2 is even, then n is even.

In Activity 38 you saw that the contrapositive of this statement is

if n is odd, then n^2 is odd.

You saw that this statement is true in Example 9. Since the contrapositive is equivalent to the original statement, this shows that the original implication is true.

Strategy:

To prove an implication 'if P , then Q ' using proof by contraposition

1. Write down the contrapositive implication
if NOT Q , then NOT P .
2. Use a direct method of proof to show that the contrapositive is true.
3. Deduce that the original implication is true.

You can practise using this strategy in the next activity.

Activity 39 Using proof by contraposition

Show that each of the following statements is true, by using proof by contraposition.

- (a) If n^2 is odd, then n is odd.
- (b) If n is odd, then n is not a multiple of 4.
- (c) If $m + n$ is even, then m and n are both even or m and n are both odd.

Recall that in Subsection 2.4 you saw that the following implication is true:

if n is odd, then n^2 is odd.

It was claimed that the converse of this implication is also true. The converse is the first statement given in Activity 39 above and so you can now deduce that the following equivalence is true:

n is odd if and only if n^2 is odd.

5.3 Proving Fermat's little theorem

In your study of number theory in Unit 3, you learned about *Fermat's little theorem*, which says that if p is a prime number, and a is an integer that is not a multiple of p , then

$$a^{p-1} \equiv 1 \pmod{p}.$$

In this subsection you'll see how proof by contradiction can be used to show that this statement is true. This proof has several stages and you would not be expected to produce such a proof on your own!

We will prove the statement in the special case where the prime p is equal to 11. Once you have seen the proof in this case, you should have a sense of how to extend it to other primes.

We must show that for any integer a that is not a multiple of 11,

$$a^{10} \equiv 1 \pmod{11}.$$

Let a be such an integer. Consider the eleven integers

$$0a, 1a, 2a, 3a, 4a, 5a, 6a, 7a, 8a, 9a, 10a.$$

(The integers $0a$ and $1a$ could be written as 0 and a , but that would obscure the pattern.) We begin by showing that no two of these eleven integers are congruent modulo 11. We'll do this using proof by contradiction. We'll suppose that two of the integers *are* congruent; that is, we'll assume that there are two different integers m and n in $\{0, 1, 2, \dots, 10\}$ for which

$$ma \equiv na \pmod{11}.$$

Since 11 is a prime number, and a is not divisible by 11, the integers a and 11 are coprime. You saw, in the unit on number theory, that this implies that a has a multiplicative inverse modulo 11. So we can multiply both sides of the congruence above by a multiplicative inverse of a modulo 11. This gives the congruence

$$m \equiv n \pmod{11}.$$

But this is impossible, since m and n are different elements of $\{0, 1, 2, \dots, 10\}$. Our assumption that two of the integers $0a, 1a, 2a, 3a, \dots, 10a$ are congruent modulo 11 must therefore be false. It follows that none of the integers $0a, 1a, 2a, 3a, \dots, 10a$ are congruent to each other modulo 11.

Since $0a = 0$, this shows that the remaining ten integers $1a, 2a, 3a, \dots, 10a$ must be congruent to the ten integers $1, 2, 3, \dots, 10$ modulo 11 in some order. (For example, if $a = 2$ then

$$\begin{aligned} a &\equiv 2 \pmod{11}, & 2a &\equiv 4 \pmod{11}, & 3a &\equiv 6 \pmod{11}, \\ 4a &\equiv 8 \pmod{11}, & 5a &\equiv 10 \pmod{11}, & 6a &\equiv 1 \pmod{11}, \\ 7a &\equiv 3 \pmod{11}, & 8a &\equiv 5 \pmod{11}, & 9a &\equiv 7 \pmod{11}, \\ 10a &\equiv 9 \pmod{11}. \end{aligned}$$

Therefore the product of the integers $1a, 2a, 3a, \dots, 10a$ is equal to the product of the integers $1, 2, 3, \dots, 10$ modulo 11. That is,

$$1a \times 2a \times 3a \times \cdots \times 10a \equiv 1 \times 2 \times 3 \times \cdots \times 10 \pmod{11}.$$

Remember that we use the notation $n!$ to represent the product

$$1 \times 2 \times 3 \times \cdots \times n,$$

which is called n **factorial**. Using this notation we can write the congruence above as

$$10! a^{10} \equiv 10! \pmod{11}.$$

Since 11 is not a factor of $10!$ (because it is prime and larger than any of the factors of $10!$), the integers $10!$ and 11 are coprime. Therefore $10!$ has a multiplicative inverse modulo 11. If we multiply both sides of the congruence above by this multiplicative inverse, then we obtain

$$a^{10} \equiv 1 \pmod{11}.$$

This is the conclusion of Fermat's little theorem in the case where $p = 11$. The reasoning for other primes p is very similar, and this proves Fermat's little theorem.

6 Summary

You have now met a variety of techniques that you can use to prove a conjecture. The following decision tree should help you to identify which technique is the most appropriate for a particular statement that you want to show to be true or false.

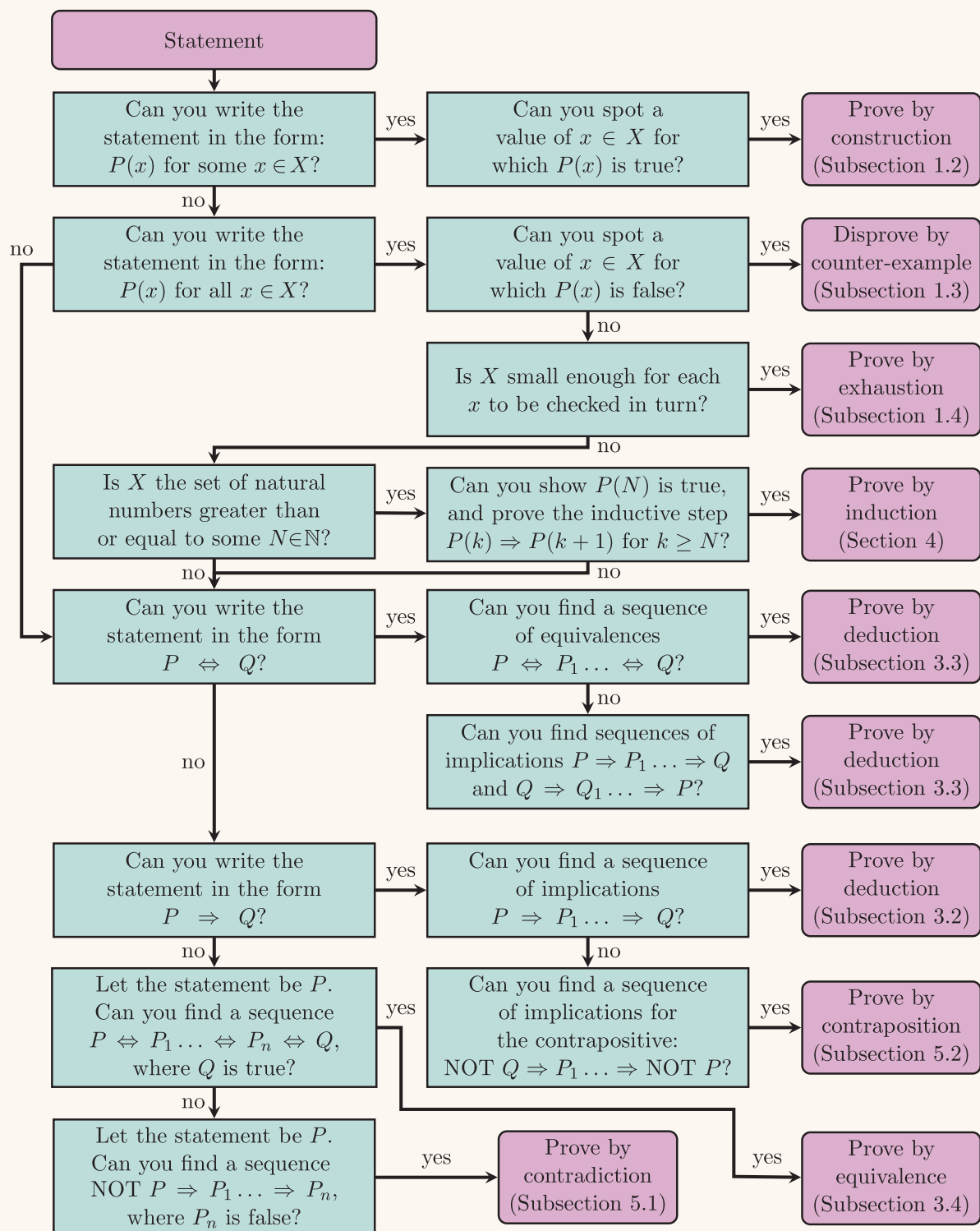


Figure 7 A decision tree to help identify a technique for proving that a statement is either true or false

Don't feel that you have to follow this decision tree in detail when proving a result. Its purpose is to draw together the main ideas that have been introduced in the unit in the hope that it will help you to focus on the kinds of questions you need to ask yourself when deciding how to approach a proof.

Activity 40 *Identifying appropriate proof techniques*

Using the decision tree above as a guide, determine whether each of the following statements is true or false.

- (a) $n! > 2^n$, for $n \geq 4$.
- (b) $x^2 > (x - 1)^2$, for all real x .
- (c) $2x^2 > (2x - 2)(x - 1)$, for $x > \frac{1}{2}$.
- (d) $2n + 1$ is prime, for $n \in \{1, 2, 3\}$.
- (e) There is no real number x such that $4x^2 + 1 < 4x$.
- (f) The function $f(x) = 2x^4 + x^2$ is one-to-one.
- (g) The product of two odd numbers is odd.

Learning outcomes

After studying this unit, you should be able to:

- understand and use the terminology associated with mathematical statements
- understand how a counter-example can be used to show that a universal statement is false
- use proof by exhaustion to prove the truth of a statement concerning a set with finitely many elements
- form the negation of a mathematical statement
- understand how the truth of a combined statement depends on the truth of the simpler statements from which it is formed
- form the converse of an implication
- understand the differences between implications and equivalences and use these correctly
- construct a proof using a sequence of deductions
- construct a proof using a sequence of equivalences
- use proof by induction to prove general statements that hold for all natural numbers
- use proof by contradiction
- form the contrapositive of an implication and understand that the contrapositive is equivalent to the original implication
- use proof by contraposition.

Solutions to activities

Solution to Activity 1

- (a) False
- (b) True
- (c) True
- (d) False

Solution to Activity 2

The primes less than 20 are: 2, 3, 5, 7, 11, 13, 17 and 19, so the following twin prime pairs are the only ones for which the smaller value is less than 20:

3 and 5, 5 and 7, 11 and 13, 17 and 19.

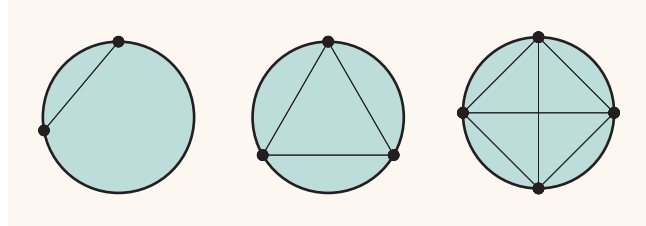
Solution to Activity 3

If $n = 1$ then no straight lines can be drawn and so the number of regions is 1.

If $n = 2$ then exactly one straight line can be drawn and so the number of regions is 2, as shown on the left below.

If $n = 3$ then three straight lines can be drawn, producing 4 regions, as shown in the middle below.

If $n = 4$ then six straight lines can be drawn, producing 8 regions, as shown on the right below.



If $n = 5$ then ten straight lines can be drawn, producing 16 regions, as shown in the diagram in the question.

At this point it might seem reasonable to conjecture that, for each $n \in \mathbb{N}$, the maximum number of regions is 2^{n-1} .

Solution to Activity 4

We'll assume n takes integer values.

- (a) There are clearly many possible solutions. For example, $P(n)$ is true for $n = 1$ and $n = 3$, and $P(n)$ is false for $n = 2$ and $n = 4$.
- (b) Again, there are many possible solutions. For example, $P(n)$ is true for $n = 5$ and $n = 60$, and $P(n)$ is false for $n = 4$ and $n = 81$.

Solution to Activity 5

- (a) This is an existential statement.
- (b) This is not an existential statement.
- (c) This is an existential statement.
- (d) This is not an existential statement.
- (e) This is an existential statement.
- (f) This is an existential statement.
- (g) This is not an existential statement.

Solution to Activity 6

- (a) This statement can be rewritten as

there exists an odd number n for which $P(n)$ is true,

where $P(n)$ is the variable proposition

n^2 is odd.

If $n = 1$, then $n^2 = 1$ is odd; that is, $P(1)$ is true.

So the statement is true. (In fact any odd number could be used here.)

- (b) This statement can be rewritten as

there exists a real number x for which $P(x)$ is true,

where $P(x)$ is the variable proposition

$x < 0$.

If $x = -1$, then $x < 0$; that is, $P(-1)$ is true.

So the statement is true. (Any negative real number could be used here.)

- (c) This statement can be rewritten as

there exists a real number x for which $P(x)$ is true,

where $P(x)$ is the variable proposition

$x^2 < 0$.

This is false as $x^2 \geq 0$ for all real values of x .

Solution to Activity 7

- (a) This is an existential statement.
- (b) This is a universal statement.
- (c) This is neither a universal statement nor an existential statement.
- (d) This is an existential statement.
- (e) This is a universal statement.
- (f) This is neither a universal statement nor an existential statement.
- (g) This is an existential statement.
- (h) This is a universal statement.
- (i) This is neither a universal statement nor an existential statement.

Solution to Activity 8

If $n = 2$, then

$$n! + 1 = 2! + 1 = 2 + 1 = 3,$$

which is prime. So 2 is not a counter-example.

If $n = 3$, then

$$n! + 1 = 3! + 1 = 6 + 1 = 7,$$

which is prime. So 3 is not a counter-example.

If $n = 5$, then

$$n! + 1 = 5! + 1 = 120 + 1 = 121.$$

But $121 = 11 \times 11$ and so 121 is not a prime number. Thus 5 is a counter-example and the statement is false.

Solution to Activity 9

In order to show that the function is not one-to-one, we must show that there are two values a and b with $a \neq b$, such that $f(a) = f(b)$. A simple counter-example is that $f(1) = f(-1) = 2$, and so the statement is false.

(In fact any choice of a and b with $b = -a$ will do, since for any number a we have $f(a) = f(-a) = a^2 + 1$.)

Solution to Activity 10

For each value of $n \in \{1, 2, 3, 4\}$, we evaluate the left- and right-hand sides of the given equation and show that both sides are equal.

If $n = 1$ then

$$1 + 3 + 5 + \cdots + (2n - 1) = 1$$

and $n^2 = 1$.

If $n = 2$ then

$$1 + 3 + 5 + \cdots + (2n - 1) = 1 + 3 = 4$$

and $n^2 = 4$.

If $n = 3$ then

$$1 + 3 + 5 + \cdots + (2n - 1) = 1 + 3 + 5 = 9$$

and $n^2 = 9$.

If $n = 4$ then

$$1 + 3 + 5 + \cdots + (2n - 1) = 1 + 3 + 5 + 7 = 16$$

and $n^2 = 16$.

Thus both sides of the equation are equal for each value of $n \in \{1, 2, 3, 4\}$ and hence the statement is true.

Solution to Activity 11

The negations are as follows.

- (a) 4 is not prime
- (b) $3 \geq 4$.
- (c) n is odd.
- (d) n is not a multiple of 5.
- (e) $x + 2 > x^2$.

Solution to Activity 12

- (a) This is a universal statement and the negation can be written as

there exists $x \in \mathbb{R}$ such that $x^2 < 0$.

The original statement is true.

- (b) This is an existential statement and the negation can be written as

every multiple of 3 is even.

The original statement is true.

- (c) This is a universal statement and the negation can be written as

there exists a prime number that is even.

The negation is true (since 2 is a prime number).

- (d) This is an existential statement and the negation can be written as

$2n + 1$ is even for all $n \in \mathbb{N}$.

The original statement is true.

- (e) This is a universal statement and the negation can be written as

there exists a natural number n such that $n^2 \geq 32n$.

The negation is true (for example, consider $n = 32$).

Solution to Activity 13

- (a) This statement is true (since both simple statements are true).
 (b) This statement is false (since both simple statements are false).
 (c) This statement is false (since the statement that 2 is odd is false).
 (d) This statement is false (since the statement that 4 is odd is false).

Solution to Activity 14

- (a) This statement is true (since both simple statements are true).
 (b) This statement is false (since both simple statements are false).
 (c) This statement is true (since the statement that 4 is even is true).
 (d) This statement is true (since the statement that 2 is even is true).

Solution to Activity 15

- (a) $P(6)$ is true and $Q(6)$ is true.
 So $P(6)$ AND $Q(6)$ is true, and $P(6)$ OR $Q(6)$ is true.
 (b) $P(7)$ is false and $Q(7)$ is false.
 So $P(7)$ AND $Q(7)$ is false, and $P(7)$ OR $Q(7)$ is false.
 (c) $P(8)$ is true and $Q(8)$ is false.
 So $P(8)$ AND $Q(8)$ is false, and $P(8)$ OR $Q(8)$ is true.

Solution to Activity 16

- (a) The table is as follows.

P	Q	P AND Q	NOT (P AND Q)
true	true	true	false
true	false	false	true
false	true	false	true
false	false	false	true

- (b) The table is as follows.

P	Q	NOT P	NOT Q	(NOT P) OR (NOT Q)
true	true	false	false	false
true	false	false	true	true
false	true	true	false	true
false	false	true	true	true

Both tables have the same final columns.

Solution to Activity 17

- (a) The table is as follows.

P	Q	P OR Q	NOT (P OR Q)
true	true	true	false
true	false	true	false
false	true	true	false
false	false	false	true

- (b) The table is as follows.

P	Q	NOT P	NOT Q	(NOT P) AND (NOT Q)
true	true	false	false	false
true	false	false	true	false
false	true	true	false	false
false	false	true	true	true

Both tables have the same final columns.

Solution to Activity 18

- (a) The negation is
1 is odd and 4 is odd.
(This can also be written as
both 1 and 4 are odd.)
- (b) The negation is
1 is odd or 4 is odd.
(This can also be written as
at least one of 1 and 4 is odd.)
- (c) The statement can be written as
 m is even or n is even.
The negation is
 m is odd and n is odd.
(This can also be written as
both m and n are odd.)
- (d) The statement can be written as
 m is even and n is even.
The negation is
 m is odd or n is odd.
(This can also be written as
at least one of m and n is odd.)

Solution to Activity 19

- (a) If $n = 2$ then n is prime but n is not odd. So 2 is a counter-example and the implication is false. (This is the only counter-example.)
- (b) There are many possible counter-examples; in fact any value of x other than $x = 0$ is a counter-example. For example, if $x = 1$, then $x \in \mathbb{R}$ but $x^2 = 1^2 = 1 > 0$. So 1 is a counter-example and the implication is false.
- (c) Any odd value of n is a counter-example. For example, if $n = 1$, then n is odd but $2n = 2 \times 1 = 2$ which is even. So 1 is a counter-example and the implication is false.
- (d) There are many possible counter-examples. For example, if $x = \frac{1}{2}$, then $x > 0$ but $x + 1 = \frac{1}{2} + 1 = \frac{3}{2} < 2$. So $\frac{1}{2}$ is a counter-example and the implication is false.
- (e) There are many possible counter-examples but it may take a while for you to find one. It may help to use the method illustrated in

Example 6. For example, if $n = 13$, then n is prime but $3n - 4 = 3 \times 13 - 4 = 35$, which is not prime. So 13 is a counter-example and the implication is false.

(You might have arrived at the counter-example 13 by first trying the smaller primes 2, 3, 5, 7 and 11, none of which are counter-examples.)

Solution to Activity 20

- (a) n is odd $\Rightarrow n^2$ is odd.
- (b) n is a multiple of 6 $\Rightarrow n$ is a multiple of 3.
- (c) $x > 0 \Rightarrow x + 1 > 1$.
- (d) $x \geq 2 \Rightarrow x^2 \geq 4$.
- (e) n is a multiple of 4 $\Rightarrow n$ is even.

Solution to Activity 21

- (a) The converse is
if $2n$ is odd, then n is odd.
The implication is false and the converse is true.
(Remember that an implication 'if P , then Q ' is always true if P is a false statement, regardless of whether Q is true or false.)
- (b) This implication can be written as
if $x > 0$, then $x + 1 > 1$.
The converse of this is
if $x + 1 > 1$, then $x > 0$.
Both the implication and the converse are true.
- (c) This implication can be written as
if n is a multiple of 6, then n is a multiple of 3.
The converse of this is
if n is a multiple of 3, then n is a multiple of 6.
The implication is true and the converse is false.
- (d) This implication can be written as
if $x^2 \geq 4$, then $x \geq 2$.
The converse of this is
if $x \geq 2$, then $x^2 \geq 4$.
The implication is false and the converse is true.

- (e) This implication can be written as
 if n is a multiple of 4, then n is even.
 The converse of this is
 if n is even, then n is a multiple of 4.
 The implication is true and the converse is false.

Solution to Activity 22

- (a) By the rules for manipulating inequalities that you met in MST124 and revised in Unit 1, this statement is true.
- (b) This statement is false since, for example, $R(-4)$ is true but $P(-4)$ is false.
- (c) This statement is false since $R(x) \Rightarrow P(x)$ is false.
- (d) By the rules for manipulating inequalities, this statement is true.
- (e) This statement is false since, for example, $R(4)$ is true but $Q(4)$ is false.
- (f) This statement is false since $R(x) \Rightarrow Q(x)$ is false.
- (g) This statement is true.
- (h) This statement is true.
- (i) This statement is true.

Solution to Activity 23

- (a) Let P and Q be the following statements:
 P means: The last bus has gone.
 Q means: You must find a taxi.
 The statements that you know to be true are P and $P \Rightarrow Q$, and the conclusion is that statement Q is true. So this deduction is valid.
- (b) Let P and Q be the following statements:
 P means: Spot is a dog.
 Q means: Spot has four legs.
 The statements that you know to be true are $P \Rightarrow Q$ and Q . The argument attempts to deduce that P is true. This is not a valid deduction. (Spot could be a cat!)

- (c) Let P and Q be the following statements:
 P means: Winston is a cat.
 Q means: Winston likes fish.
 The statements that you know to be true are P and $P \Rightarrow Q$, and the conclusion is that statement Q is true. So this deduction is valid.

- (d) Let P and Q be the following statements:
 P means: It is raining.
 Q means: The children play indoors.
 You know that $P \Rightarrow Q$ is true and that P is false. You attempt to deduce that Q is false. This is not a valid deduction. (The children might play indoors if the weather is too hot!)

- (e) Let P and Q be the following statements:
 P means: Dolly is a sheep.
 Q means: Dolly is green.
 The statements that you know to be true are P and $P \Rightarrow Q$, and the conclusion is that statement Q is true. So this deduction is valid (even though it is nonsensical!).

Solution to Activity 24

- (a) Let $P(n)$ and $Q(n)$ be the following statements:
 $P(n)$ means: n is odd.
 $Q(n)$ means: $n = 2k + 1$ for some integer k .
 The statements that you know to be true are P and $P \Rightarrow Q$, and the conclusion is that statement Q is true. So this deduction is valid.
- (b) Let $P(x)$ and $Q(x)$ be the following statements:
 $P(x)$ means: $x > 2$.
 $Q(x)$ means: $x^2 > 4$.
 You know that $P(x) \Rightarrow Q(x)$ is true and that $P(x)$ is false. The argument attempts to deduce that $Q(x)$ is false. This is not a valid deduction; for example, $P(-3)$ is false but $Q(-3)$ is true.
- (c) Let $P(n)$ and $Q(n)$ be the following statements:
 $P(n)$ means: n is a multiple of 4.
 $Q(n)$ means: n is even.
 The statements that you know to be true are $Q(n)$ and $P(n) \Rightarrow Q(n)$. The argument attempts to deduce that $P(n)$ is true. This is not a valid deduction; for example, $Q(2)$ is true but $P(2)$ is false.

Solution to Activity 25

The error occurred when the equation was divided through by $x - 1$. The implication

$$(x + 1)(x - 1) = x - 1 \Rightarrow x + 1 = 1$$

is valid only if $x - 1 \neq 0$, but here $x - 1$ is equal to zero, since the original assumption was that $x = 1$.

(It is never valid to divide through by zero in this way and this is a common mistake that often appears in false proofs.)

Solution to Activity 26

(a) Let n be an even integer. Then

$$n = 2k \text{ for some integer } k$$

and so

$$n^2 = 4k^2 = 2(2k^2).$$

Thus n^2 is even, since $2k^2$ is an integer.

(b) Let m be an odd integer and n be an even integer. Then

$$m = 2k + 1 \text{ and } n = 2l$$

for some integers k and l . Thus

$$\begin{aligned} m + n &= 2k + 1 + 2l \\ &= 2(k + l) + 1. \end{aligned}$$

Thus $m + n$ is odd, since $k + l$ is an integer.

Solution to Activity 27

This statement can be written as

if n is a multiple of 4, then n is even.

Let n be a multiple of 4. Then

$$n = 4k \text{ for some integer } k.$$

Hence

$$n = 2(2k) \text{ for some integer } k.$$

Thus n is even.

Solution to Activity 28

We must show that, for all real numbers x_1 and x_2 , if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Let x_1 and x_2 be real numbers and suppose that $f(x_1) = f(x_2)$. Then

$$4x_1 + 3 = 4x_2 + 3$$

and hence

$$4x_1 = 4x_2.$$

Thus $x_1 = x_2$ and so f is one-to-one.

Solution to Activity 29

Let n be an integer such that $n + 2$ is odd. Then

$$n + 2 = 2k + 1$$

for some integer k . Hence

$$n = 2k - 1.$$

So

$$\begin{aligned} 5n - 1 &= 5(2k - 1) - 1 \\ &= 10k - 6 \\ &= 2(5k - 3). \end{aligned}$$

Thus $5n - 1$ is even, since $5k - 3$ is an integer.

Now let n be an integer such that $5n - 1$ is even.

Then

$$5n - 1 = 2k$$

for some integer k . Subtracting $4n$ and adding 3 on both sides gives

$$\begin{aligned} n + 2 &= 2k - 4n + 3 \\ &= 2(k - 2n + 1) + 1. \end{aligned}$$

So $n + 2$ is odd, since $k - 2n + 1$ is an integer.

Solution to Activity 30

$$2x + 3y = 6$$

$$\Leftrightarrow 3y = 6 - 2x$$

$$\Leftrightarrow y = 2 - \frac{2}{3}x.$$

Solution to Activity 31

(a) Let x be real. Then

$$\begin{aligned} x^2 &\geq (x + 2)(x - 2) \\ \Leftrightarrow x^2 &\geq x^2 - 4 \\ \Leftrightarrow 4 &\geq 0. \end{aligned}$$

This last inequality is clearly true and so the first inequality is also true.

(b) Let $n \geq 2$. Then

$$\begin{aligned} (2n + 2)(5n + 1) &> (3n + 2)^2 \\ \Leftrightarrow 10n^2 + 12n + 2 &> 9n^2 + 12n + 4 \\ \Leftrightarrow n^2 &> 2. \end{aligned}$$

This last inequality is clearly true since $n \geq 2$ and so the first inequality is also true.

Solution to Activity 32

(a) Let $P(n)$ be the statement

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Then $P(1)$ is true, because $1 = 1^2$.

Now let $k \geq 1$, and assume that $P(k)$ is true; that is,

$$1 + 3 + \cdots + (2k - 1) = k^2.$$

We want to deduce that $P(k + 1)$ is true; that is,

$$1 + 3 + \cdots + (2k + 1) = (k + 1)^2.$$

Now

$$\begin{aligned} 1 + 3 + \cdots + (2k + 1) &= (1 + 3 + \cdots + (2k - 1)) + (2k + 1) \\ &= k^2 + (2k + 1) \quad (\text{by } P(k)) \\ &= (k + 1)^2. \end{aligned}$$

That is, $P(k + 1)$ is true.

Thus $P(k + 1)$ is true if $P(k)$ is true; that is,

$$P(k) \Rightarrow P(k + 1), \text{ for } k = 1, 2, \dots$$

Hence, by mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(b) Let $P(n)$ be the statement

$$1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2.$$

Then $P(1)$ is true, because

$$1^3 = 1 \quad \text{and} \quad \frac{1}{4}(1)(1 + 1)^2 = 1.$$

Now let $k \geq 1$, and assume that $P(k)$ is true; that is,

$$1^3 + 2^3 + \cdots + k^3 = \frac{1}{4}k^2(k + 1)^2.$$

We want to deduce that $P(k + 1)$ is true; that is,

$$1^3 + 2^3 + \cdots + (k + 1)^3 = \frac{1}{4}(k + 1)^2(k + 2)^2.$$

Now

$$\begin{aligned} 1^3 + 2^3 + \cdots + (k + 1)^3 &= (1^3 + 2^3 + \cdots + k^3) + (k + 1)^3 \\ &= \frac{1}{4}k^2(k + 1)^2 + (k + 1)^3 \quad (\text{by } P(k)) \\ &= \frac{1}{4}(k + 1)^2(k^2 + 4(k + 1)) \\ &= \frac{1}{4}(k + 1)^2(k^2 + 4k + 4) \\ &= \frac{1}{4}(k + 1)^2(k + 2)^2. \end{aligned}$$

Thus

$$P(k) \Rightarrow P(k + 1), \text{ for } k = 1, 2, \dots$$

Hence, by mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Solution to Activity 33

Let $P(n)$ be the statement

$$f^{(n)}(x) = (n + x)e^x,$$

where $f(x) = xe^x$.

Then $P(1)$ is true, because, by the product rule,

$$f'(x) = e^x + xe^x = (1 + x)e^x.$$

Now let $k \geq 1$, and assume that $P(k)$ is true; that is,

$$f^{(k)}(x) = (k + x)e^x.$$

We want to deduce that $P(k + 1)$ is true; that is,

$$f^{(k+1)}(x) = (k + 1 + x)e^x.$$

To do this, we differentiate $f^{(k)}(x)$. This gives

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}(f^{(k)}(x)) \\ &= \frac{d}{dx}((k + x)e^x) \quad (\text{by } P(k)) \\ &= e^x + (k + x)e^x \quad (\text{by the product rule}) \\ &= (k + 1 + x)e^x. \end{aligned}$$

Thus

$$P(k) \Rightarrow P(k + 1), \text{ for } k = 1, 2, \dots$$

Hence, by mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Solution to Activity 34

When $n = 2$,

$$2^n = 2^2 = 4 \text{ and } 4n = 4 \times 2 = 8,$$

so $P(2)$ is false.

When $n = 3$,

$$2^n = 2^3 = 8 \text{ and } 4n = 4 \times 3 = 12,$$

so $P(3)$ is false.

When $n = 4$,

$$2^n = 2^4 = 16 \text{ and } 4n = 4 \times 4 = 16,$$

so $P(4)$ is false.

When $n = 5$,

$$2^n = 2^5 = 32 \text{ and } 4n = 4 \times 5 = 20,$$

so $P(5)$ is true.

When $n = 6$,

$$2^n = 2^6 = 64 \text{ and } 4n = 4 \times 6 = 24,$$

so $P(6)$ is true.

Solution to Activity 35

- (a) Let $P(n)$ be the statement
 $3^n > 10n$.

Then $P(4)$ is true, because

$$3^4 = 81 > 40 = 10 \times 4.$$

Now let $k \geq 4$, and assume that $P(k)$ is true; that is, $3^k > 10k$. We want to deduce that $P(k+1)$ is true; that is,

$$3^{k+1} > 10(k+1).$$

Now

$$\begin{aligned} 3^{k+1} &= 3 \times 3^k \\ &> 3 \times 10k = 30k, \quad (\text{by } P(k)). \end{aligned}$$

We would like to show that $30k \geq 10(k+1)$.

Now

$$\begin{aligned} 30k &\geq 10(k+1) \\ \Leftrightarrow 30k &\geq 10k + 10 \\ \Leftrightarrow 20k &\geq 10. \end{aligned}$$

This last inequality is clearly true for $k \geq 4$ and hence $30k \geq 10(k+1)$. Since we have already shown that $3^{k+1} > 30k$, it follows that

$$3^{k+1} > 10(k+1),$$

as required. Thus

$$P(k) \Rightarrow P(k+1), \text{ for } k \geq 4.$$

Hence, by mathematical induction, $P(n)$ is true, for $n \geq 4$.

- (b) Let $P(n)$ be the statement
 $n! > 3^n$.

Then $P(7)$ is true, because

$$7! = 5040 > 2187 = 3^7.$$

Now let $k \geq 7$, and assume that $P(k)$ is true; that is, $k! > 3^k$. We want to deduce that $P(k+1)$ is true; that is,

$$(k+1)! > 3^{k+1}.$$

Now

$$\begin{aligned} (k+1)! &= (k+1) \times k! \\ &> (k+1) \times 3^k, \quad (\text{by } P(k)). \end{aligned}$$

We would like to show that $(k+1) \times 3^k \geq 3^{k+1}$.

Now $3^{k+1} = 3 \times 3^k$ and so

$$\begin{aligned} (k+1) \times 3^k &\geq 3^{k+1} \\ \Leftrightarrow k+1 &\geq 3. \end{aligned}$$

This last inequality is clearly true for $k \geq 7$ and hence $(k+1) \times 3^k \geq 3^{k+1}$. Since we have

already shown that

$$(k+1)! > (k+1) \times 3^k,$$

it follows that

$$(k+1)! > 3^{k+1},$$

as required. Thus

$$P(k) \Rightarrow P(k+1), \text{ for } k \geq 7.$$

Hence, by mathematical induction, $P(n)$ is true, for $n \geq 7$.

Solution to Activity 36

- (a) Assume that this statement is not true; that is, assume that there exist integers m and n such that

$$5m + 10n = 549.$$

The left-hand side of this equation is divisible by 5, since $5m + 10n = 5(m + 2n)$. Since the right-hand side is not divisible by 5, this gives a contradiction. Thus our assumption that the original statement was false is incorrect. This shows that the statement is true.

- (b) Assume that this statement is not true; that is, assume that there exists a real number x such that

$$x^2 + 1 < 2x.$$

Then

$$x^2 + 1 - 2x < 0;$$

that is,

$$(x-1)^2 < 0.$$

This, however, is a contradiction and so our assumption that the original statement was false is incorrect. This shows that the statement is true.

- (c) Assume that this statement is not true; that is, assume that there exists an integer n such that

$$n^2 < (n+1)(n-1).$$

Then

$$n^2 < n^2 - 1;$$

that is,

$$0 < -1.$$

This, however, is a contradiction and so our assumption that the original statement was false is incorrect. This shows that the statement is true.

Solution to Activity 37

Assume that this is not true; in other words, assume that $\sqrt{5}$ is rational.

Then there exist integers m and n with no common divisors such that $m/n = \sqrt{5}$.

Then $m^2/n^2 = 5$; that is,

$$m^2 = 5n^2.$$

This implies that 5 is a divisor of m^2 . Hence, since 5 is prime, 5 is a divisor of m . Thus $m = 5k$ for some integer k . So

$$25k^2 = 5n^2$$

and hence

$$5k^2 = n^2.$$

In the same way, we can now deduce that 5 is a divisor of n . This, however, is a contradiction, since m and n have no common divisors. Thus our assumption that $\sqrt{5}$ is rational was false and hence $\sqrt{5}$ is irrational, as claimed.

Solution to Activity 38

(a) The contrapositive is

if n is not even, then n^2 is not even,

or, more simply,

if n is odd, then n^2 is odd.

(b) The contrapositive is

if n is not odd, then n^2 is not odd,

or, more simply,

if n is even, then n^2 is even.

(c) The contrapositive is

if n is a multiple of 4, then n is not odd,

or, more simply,

if n is a multiple of 4, then n is even.

(d) The contrapositive is

if $x + 1 \leq 1$, then $x \leq 0$.

Solution to Activity 39

(a) In Activity 38(b) you saw that the contrapositive is

if n is even, then n^2 is even.

We now show that this is true. Let n be even.

Then $n = 2k$ for some integer k and so

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Thus n^2 is even, since $2k^2$ is an integer. So the contrapositive is true and hence the original statement is true.

(b) In Activity 38(c) you saw that the contrapositive is

if n is a multiple of 4, then n is even.

Also, in Activity 27, you saw that the contrapositive is true. Thus the original statement is true.

(c) The contrapositive is

if one of m and n is even and the other is odd, then $m + n$ is odd.

We now show that this is true. Let one of m and n be even and the other be odd. Then one of them is equal to $2k$ and the other is equal to $2l + 1$, for some integers k and l . Thus

$$m + n = 2k + 2l + 1 = 2(k + l) + 1.$$

Thus $m + n$ is odd, since $k + l$ is an integer. So the contrapositive is true and hence the original statement is true.

Solution to Activity 40

(a) Let $P(n)$ be the statement

$$n! > 2^n.$$

The decision tree indicates that it seems sensible to try using proof by induction.

We begin by noting that $P(4)$ is true, since $4! = 24 > 16 = 2^4$.

Now let $k \geq 4$, and assume that $P(k)$ is true; that is, $k! > 2^k$. We want to deduce that $P(k + 1)$ is true; that is, $(k + 1)! > 2^{k+1}$. Now

$$\begin{aligned} (k + 1)! &= (k + 1) \times k! \\ &> (k + 1) \times 2^k \quad (\text{by } P(k)) \\ &> 2 \times 2^k \\ &= 2^{k+1}. \end{aligned}$$

So $(k+1)! > 2^{k+1}$ as required. Thus

$$P(k) \Rightarrow P(k+1), \text{ for } k \geq 4.$$

Hence, by mathematical induction, $P(n)$ is true for $n \geq 4$.

- (b) You need to be careful with this statement. You might be tempted to think that it is true, since it is always true that $x > x-1$. It is not, however, valid to square both sides of this inequality. If you think about it more carefully, then you might begin to spot values of x for which the inequality is not satisfied. In other words, you can find a counter-example. There are many possibilities here; for example, if $x = 0$, then $x^2 = 0$ and $(x-1)^2 = 1 > 0$, so 0 is a counter-example and the statement is false.

- (c) You may not be sure whether this statement is true or false. If you try a few values of x then you will find that the statement is true for these values of x . It seems sensible to try to prove the statement by rearranging the inequality. In other words, you attempt to use a sequence of equivalences to produce a statement that you know to be true.

Let $x > \frac{1}{2}$. Then

$$\begin{aligned} 2x^2 &> (2x-2)(x-1) \\ \Leftrightarrow 2x^2 &> 2x^2 - 4x + 2 \\ \Leftrightarrow 4x &> 2 \\ \Leftrightarrow x &> \frac{1}{2}. \end{aligned}$$

This last inequality is true and so the first inequality is also true.

- (d) This statement is a universal statement concerning a set with only a small number of elements, so we can test whether the statement is true for each value in turn.

If $n = 1$, then $2n + 1 = 3$ which is prime.

If $n = 2$, then $2n + 1 = 5$ which is prime.

If $n = 3$, then $2n + 1 = 7$ which is prime.

Thus the statement is true.

- (e) If you test a few values of x to see whether you can find one such that $4x^2 + 1 < 4x$, then you should fail to find one. This suggests that the statement might be true. There are various ways of proving this; one way is to use proof by contradiction.

Suppose that the statement is not true; that is, suppose that there exists a real number x such that

$$4x^2 + 1 < 4x.$$

Then

$$4x^2 + 1 - 4x < 0;$$

that is,

$$(2x-1)^2 < 0.$$

This, however, is a contradiction and so our assumption that the original statement was false was incorrect. This shows that the statement is true.

- (f) You might guess that this statement is false, since $x^2 = (-x)^2$ and $x^4 = (-x)^4$ for any number x . Remember that you saw in Subsection 1.3 that a statement of the form ‘the function f is one-to-one’ can be written as a universal statement. So, following the decision tree, we try to find a counter-example. We must show that there are two values a and b with $a \neq b$ such that $f(a) = f(b)$. Any choice of a and b with $b = -a$ will do. For example, $f(1) = f(-1) = 3$ and so the statement is false.

- (g) You might guess that this statement is true (you might find it helpful to try a few numbers). We are asked to prove the implication

if m and n are odd numbers, then mn is also odd.

So we try to produce a proof using a sequence of deductions.

Let m and n be odd numbers. Then there exist integers k and l such that

$$m = 2k + 1 \text{ and } n = 2l + 1.$$

Then

$$\begin{aligned} mn &= (2k+1)(2l+1) \\ &= 4kl + 2k + 2l + 1 \\ &= 2(2kl + k + l) + 1. \end{aligned}$$

Thus mn is odd, since $2kl + k + l$ is an integer.

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